Chapter R

Review of Fundamentals

Functions are to calculus as numbers are to algebra.

Success in the study of calculus depends upon a solid understanding of algebra and analytic geometry. In this chapter, we review topics from these subjects which are particularly important for calculus.

R.1 Basic Algebra: Real Numbers and Inequalities

The real numbers are ordered like the points on a line.

The most important facts about the real numbers concern algebraic operations (addition, multiplication, subtraction, and division) and order (greater than and less than). In this section, we review some of these facts.

The positive whole numbers $1, 2, 3, 4, \ldots$ ($\ldots$ means "and so on") that arise from the counting process are called the *natural numbers*. The arithmetic operations of addition and multiplication can be performed within the natural numbers, but the "inverse" operations of subtraction and division lead to the introduction of zero ($3 - 3 = 0$), negative numbers ($2 - 6 = -4$), and fractions ($3 \div 5 = \frac{3}{5}$). The whole numbers, positive, zero, and negative, are called *integers*. All numbers which can be put in the form m/n, where m and n are integers, are called *rational numbers*.

Example 1 Determine whether or not the following numbers are natural numbers, integers or rational numbers.

(a) 0 (b) $3 - \dfrac{4+5}{2}$ (c) $7 - 6$ (d) $\dfrac{4+5}{-3}$

Solution

(a) 0 is not a natural number (they are only the numbers $1, 2, 3, \ldots$), but it is an integer, and so it is a rational number—every integer m is rational since it can be written as $m/1$.

(b) $3 - (4 + 5)/2 = 3 - 9/2 = -3/2$ is not a natural number or an integer, but it is rational.

(c) $7 - 6 = 1$ is a natural number, an integer, and a rational number.

(d) $(4 + 5)/(-3) = -3$ is not a natural number, but is an integer and a rational number. ▲

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The ancient Greeks already knew that lines in simple geometric figures could have lengths which did not correspond to ratios of whole numbers. For instance, the length $\sqrt{2}$ of the diagonal of a square with sides of unit length cannot be expressed in the form m/n, with m and n integers. (The same turns out to be true of π, the circumference of a circle with unit diameter.) Numbers which are not ratios of integers are called *irrational numbers*. These, together with the rational numbers, comprise the *real numbers*.

The usual arithmetic operations of addition, multiplication, subtraction, and division (except by zero) may be performed on real numbers, and these operations satisfy the usual algebraic rules. You should be familiar with these rules; some examples are "if equals are added to equals, the results are equal," "$a + b = b + a$," and "if $ab = ac$ and $a \neq 0$, we can divide both sides by a to conclude that $b = c$."

For example, to solve the equation $3x + 2 = 8$ for x, subtract 2 from both sides of the equation:

$$3x + 2 - 2 = 8 - 2,$$
$$3x = 6.$$

Dividing both sides by 3 gives $x = 2$.

Two fundamental identities from algebra that will be useful for us are

$$(a + b)(a - b) = a^2 - b^2 \quad \text{and} \quad (a + b)^2 = a^2 + 2ab + b^2.$$

Example 2 Simplify: $(a + b)(a - b) + b^2$.

Solution Since $(a + b)(a - b) = a^2 - b^2$, we have $(a + b)(a - b) + b^2 = a^2 - b^2 + b^2 = a^2$. ▲

Example 3 Expand: $(a + b)^3$.

Solution We have $(a + b)^2 = a^2 + 2ab + b^2$. Therefore

$$\begin{aligned}(a + b)^3 &= (a + b)^2(a + b) \\ &= (a^2 + 2ab + b^2)(a + b) \\ &= (a^2 + 2ab + b^2)a + (a^2 + 2ab + b^2)b \\ &= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3 \\ &= a^3 + 3a^2b + 3ab^2 + b^3.\end{aligned}$$ ▲

Another important algebraic operation is *factoring*. We try to reverse the process of expanding: $(x + r)(x + s) = x^2 + (r + s)x + rs$.

Example 4 Factor: $2x^2 + 4x - 6$.

Solution We notice first that $2x^2 + 4x - 6 = 2(x^2 + 2x - 3)$. Using the fact that the only integer factors of -3 are ± 1 and ± 3, we find by trial and error that $x^2 + 2x - 3 = (x + 3)(x - 1)$, so we have $2x^2 + 4x - 6 = 2(x + 3)(x - 1)$. ▲

The *quadratic formula* is used to solve for x in equations of the form $ax^2 + bx + c = 0$ when the left-hand side cannot be readily factored. The method of *completing the square*, by which the quadratic formula may be derived, is often more important than the formula itself.

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Example 5 Solve the equation $x^2 - 5x + 3 = 0$ by completing the square.

Solution We transform the equation by adding and subtracting $(\frac{5}{2})^2$ on the left-hand side:

$$x^2 - 5x + (\tfrac{5}{2})^2 - (\tfrac{5}{2})^2 + 3 = 0,$$

$$(x - \tfrac{5}{2})^2 - \tfrac{13}{4} = 0,$$

$$(x - \tfrac{5}{2})^2 = \tfrac{13}{4},$$

$$x - \frac{5}{2} = \pm \frac{\sqrt{13}}{2},$$

$$x = \frac{5}{2} \pm \frac{\sqrt{13}}{2}. \blacktriangle$$

Another method for completing the square is to write

$$x^2 - 5x + 3 = (x + p)^2 + q.$$

and then expand to

$$x^2 - 5x + 3 = x^2 + 2px + p^2 + q.$$

Equating coefficients, we see that $p = -\frac{5}{2}$ and $q = 3 - p^2 = 3 - \frac{25}{4} = -\frac{13}{4}$, so $x^2 - 5x + 3 = (x - \frac{5}{2})^2 - \frac{13}{4}$. This can be used to solve $x^2 - 5x + 3 = 0$ as above.

Completing the Square

To complete the square in the expression $ax^2 + bx + c$, factor out a and then add and subtract $(b/2a)^2$:

$$ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right].$$

When the method of completing the square is applied to the *general* quadratic equation $ax^2 + bx + c = 0$, one obtains the following general formula for the solution of the equation. (See Exercise 53).

Quadratic Formula

To solve $ax^2 + bx + c = 0$, where $a \neq 0$, compute

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 - 4ac > 0$, there are two solutions.
If $b^2 - 4ac = 0$, there is one solution.
If $b^2 - 4ac < 0$, there are no solutions.
(The expression $b^2 - 4ac$ is called the *discriminant*.)

In case $b^2 - 4ac < 0$, there is no real number $\sqrt{b^2 - 4ac}$, because the square of every real number is greater than or equal to zero. (Square roots of negative numbers can be found if we extend the real-number system to

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encompass the so-called *imaginary numbers*.)[1] Thus the symbol $\sqrt{r}$ represents a real number only when $r \geqslant 0$, in which case we always take $\sqrt{r}$ to mean the *non-negative* number whose square is r.

Example 6 Solve for x: (a) $4x^2 = 2x + 5$ and (b) $2x^2 + 4x - 6 = 0$.

Solution (a) Subtracting $2x+5$ from both sides of the equation gives $4x^2 - 2x - 5 = 0$, which is in the form $ax^2 + bx + c = 0$ with $a = 4$, $b = -2$, and $c = -5$. The quadratic formula gives the two roots

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(4)(-5)}}{2(4)}$$

$$= \frac{2 \pm \sqrt{4 + 80}}{8} = \frac{2 \pm \sqrt{84}}{8} = \frac{1}{4} \pm \frac{\sqrt{21 \cdot 4}}{8} = \frac{1}{4} \pm \frac{\sqrt{21}}{4}.$$

(b) An alternative to the quadratic formula is to factor. From Example 4, $2x^2 + 4x - 6 = 2(x + 3)(x - 1)$. Thus the two roots are $x = -3$ and $x = 1$. The reader may check that the quadratic formula gives the same roots. ▲

Example 7 Solve for x: $x^2 - 5x + 20 = 0$.

Solution We use the quadratic formula:

$$x = \frac{5 \pm \sqrt{25 - 4 \cdot 20}}{2}.$$

The discriminant is negative, so there are no real solutions. ▲

The real numbers have a relation of order: if two real numbers are unequal, one of them is less than the other. We may represent the real numbers as points on a line, with larger numbers to the right, as shown in Fig. R.1.1. If the number a is less than b, we write $a < b$. In this case, we also say that b is greater than a and write $b > a$.

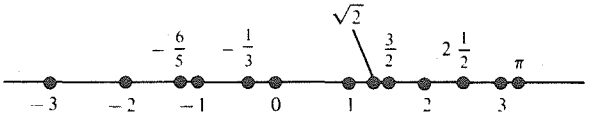

Figure R.1.1. The real number line.

Given any two numbers, a and b, exactly one of the following three possibilities holds:

(1) $a < b$,
(2) $a = b$,
(3) $a > b$.

Combinations of these possibilities have special names and notations.

If (1) or (2) holds, we write $a \leqslant b$ and say that "a is less than or equal to b."

If (2) or (3) holds, we write $a \geqslant b$ and say that "a is greater than or equal to b."

If (1) or (3) holds, we write $a \neq b$ and say that "a is unequal to b."

For example, $3 \leqslant 3$ is true, $(-2)^2 \leqslant 0$ is false (since $(-2)^2 = 4 > 0$) and $-\pi < -\frac{1}{2}\pi$ is true; note that $-\pi$ and $-\frac{1}{2}\pi$ both lie to the left of zero on the

[1] Imaginary numbers are discussed in Section 12.6.

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number line and since $-\frac{1}{2}\pi$ is only half as far from zero as $-\pi$, it lies to the right of $-\pi$.

If x is any real number, we know that $x^2 \geqslant 0$. If $x \neq 0$, we can make the stronger statement that $x^2 > 0$.

Example 8 Write the proper inequality sign between each of the following pairs of numbers:

(a) 0.0000025 and $-100{,}000$ (b) $\frac{3}{4}$ and $\frac{6}{7}$ (c) $\sqrt{12}$ and 4

Solution (a) $0.0000025 > -100{,}000$ since a positive number is always to the right of a negative number.

(b) $\frac{3}{4} < \frac{6}{7}$ since $\frac{3}{4} = \frac{21}{28}$ and $\frac{6}{7} = \frac{24}{28}$.

(c) $\sqrt{12} < 4$ since $12 < 4^2$. ▲

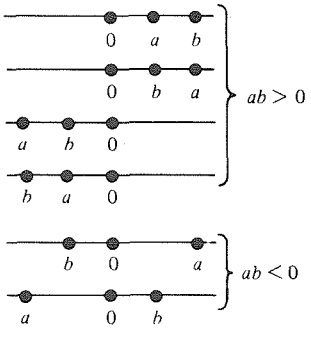

Figure R.1.2. Possible positions of a and b when they have the same sign ($ab > 0$) or opposite signs ($ab < 0$).

We can summarize the most important properties of inequalities as follows.

1. If $a < b$ and $b < c$, then $a < c$.
2. If $a < b$, then $a + c < b + c$ for any c, and $ac < bc$ if $c > 0$, while $ac > bc$ if $c < 0$. (Multiplication by a negative number reverses the sign of inequality. For instance, $3 < 4$, and multiplication by -2 gives $-6 > -8$.)
3. $ab > 0$ when a and b have the same sign; $ab < 0$ when a and b have opposite signs. (See Fig. R.1.2.)
4. If a and b are any two numbers, then $a < b$ when $a - b < 0$ and $a > b$ when $a - b > 0$.

Example 9 Transform $a + (b - c) > b - a$ to an inequality with a alone on one side.

Solution We transform by reversible steps:

$$a + b - c > b - a,$$

$$2a + b - c > b \qquad \text{(add } a \text{ to both sides)},$$

$$2a - c > 0 \qquad \text{(add } -b \text{ to both sides)},$$

$$2a > c \qquad \text{(add } c \text{ to both sides)},$$

$$a > \tfrac{1}{2}c \qquad \text{(multiply both sides by } \tfrac{1}{2}\text{)}. \blacktriangle$$

Example 10 (a) Find all numbers x for which $x^2 < 9$.

(b) Find all numbers x such that $x^2 - 2x - 3 > 0$.

Solution (a) We transform the inequality as follows (all steps are reversible):

$$x^2 < 9,$$

$$x^2 - 9 < 0, \qquad \text{(add } -9 \text{ to both sides)},$$

$$(x + 3)(x - 3) < 0 \qquad \text{(factor)}.$$

Since the product $(x + 3)(x - 3)$ is negative, the factors $x + 3$ and $x - 3$ must have opposite signs. Thus, either $x + 3 > 0$ and $x - 3 < 0$, so that $x > -3$ and $x < 3$ (that is, $-3 < x < 3$); or $x + 3 < 0$ and $x - 3 > 0$, in which case $x < -3$ and $x > 3$, which is impossible. We conclude that $x^2 < 9$ if and only if $-3 < x < 3$.

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(b) The inequality $x^2 - 2x - 3 > 0$ is the same as $(x - 3)(x + 1) > 0$. That is, $x - 3$ and $x + 1$ have the same sign. There are two cases to consider:

Case 1: $x - 3$ and $x + 1$ are both positive; that is, $x - 3 > 0$ and $x + 1 > 0$; that is, $x > 3$ and $x > -1$, which is the same as $x > 3$ (since any number greater than 3 is certainly greater than -1).

Case 2: $x - 3 < 0$ and $x + 1 < 0$; that is, $x < 3$ and $x < -1$, which is the same as $x < -1$.

Thus $x^2 - 2x - 3 > 0$ whenever $x > 3$ or $x < -1$. These numbers x are illustrated in Fig. R.1.3. (The open dot indicates that this point is not included in the shaded region—if it were included, we would have used a solid dot.) ▲

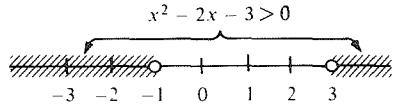

Figure R.1.3. Solution of the inequality $x^2 - 2x - 3 > 0$.

Exercises for Section R.1

In Exercises 1–4, determine whether or not each given number is a natural number, an integer, or a rational number.

1. $\frac{8}{6} - \frac{9}{4}$
2. $(-1) \div (-1)$
3. $\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right)$
4. $\pi - \frac{1}{2}$

Simplify the expressions in Exercises 5–8.

5. $(a - 3)(b + c) - (ac + 2b)$,
6. $(b^2 - a)^2 - a(2b^2 - a)$
7. $a^2c + (a - b)(b - c)(a + b)$
8. $(3a + 2)^2 - (4a + b)(2a - 1)$

Expand the expressions in Exercises 9–12.

9. $(a - b)^3$
10. $(3a + b^2 + c)^2$
11. $(b + c)^4$
12. $(2c - b)^2(2c + b)^2$

Factor the expressions in Exercises 13–20.

13. $x^2 + 5x + 6$
14. $x^2 - 5x + 6$
15. $x^2 - 5x - 6$
16. $x^2 + 5x - 6$
17. $3x^2 - 6x - 24$
18. $-5x^2 + 15x - 10$
19. $x^2 - 1$
20. $4x^2 - 9$

Solve for x in Exercises 21–24.

21. $2(3x - 7) - (4x - 10) = 0$
22. $3(3 + 2x) + (2x - 1) = 8$
23. $(2x + 1)^2 + (9 - 4x^2) + (x - 5) = 10$
24. $8(x + 1)^2 - 8x + 10 = 0$
25. Verify that $x^3 - 1 = (x - 1)(x^2 + x + 1)$.
26. Factor $x^3 + 1$ into linear and quadratic factors.
27. Factor $x^3 + x^2 - 2x$ into linear factors.
28. Factor $x^4 - 2x^2 + 1$ into linear factors. [*Hint:* First consider x^2 as the variable.]

Solve the equations in Exercises 29 and 30 in three ways: (a) by factoring; (b) by completing the square; (c) by using the quadratic formula.

29. $x^2 + 5x + 4 = 0$
30. $4x^2 - 12x + 9 = 0$

Solve for x in Exercises 31–36.

31. $x^2 + \frac{1}{2}x - \frac{1}{2} = 0$
32. $4x^2 - 18x + 20 = 0$
33. $-x^2 + 5x + 0.3 = 0$
34. $5x^2 + 2x - 1 = 0$
35. $x^2 - 5x + 7 = 0$
36. $0.1x^2 - 1.3x + 0.7 = 0$

Solve for x in Exercises 37–42.

37. $x^2 + 4 = 3x^2 - x$
38. $4x = 3x^2 + 7$
39. $2x + x^2 = 9 + x^2$
40. $(5 - x)(2 - x) = 1$
41. $2x^2 - 2\sqrt{7}\,x + \frac{7}{2} = 0$
42. $x^2 + 9x = 0$
43. Put the following list of numbers in ascending order. (Try to do it without finding decimal equivalents for the numbers.)

$$-\tfrac{5}{3},\quad -\sqrt{2},\quad -\tfrac{7}{5},\quad \tfrac{22}{7},\quad 3,\quad \tfrac{23}{8},\quad 0,\quad \tfrac{9}{5},\quad -\sqrt{3}$$

44. Put the following numbers in ascending order. (Do not use a calculator):

$$-\sqrt{90},\quad 8,\quad -\tfrac{81}{8},\quad \tfrac{81}{10},\quad \sqrt{90},\quad 9,\quad -9,\quad -8$$

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Simplify the inequalities in Exercises 45–50.

45. $(a-b)+c>2c-b$
46. $(a+c^2)+c(a-c)\geqslant ac+1$
47. $ab-(a-2b)b<b^2+c$
48. $2(a+ac)-4ac>2a-c$
49. $b(b+2)>(b+1)(b+2)$
50. $(a-b)^2>3-2ab+a^2$
51. Find all numbers x such that: (a) $4x-13<3$, (b) $2(7-x)\geqslant x+1$, (c) $5(x-3)-2x+6>0$. Sketch your solutions on a number line.
52. Find all numbers x such that: (a) $2(x^2-x)>0$, (b) $3x^2+2x-1\geqslant 0$, (c) $x^2-5x+6<0$. Sketch your solutions on a number line.

★53. (a) Prove the quadratic formula by the method of completing the square. (b) Show that the equation $ax^2+bx+c=0$, where $a\neq 0$, has two equal roots if and only if $b^2=4ac$.

R.2 Intervals and Absolute Values

The number x belongs to the interval $[a-r,a+r]$ when $|x-a|\leqslant r$.

In this section, two important notations are discussed. The first is that of intervals on the real-number line, and the second is the absolute value $|x|$, which is the distance from the origin (zero) to x.

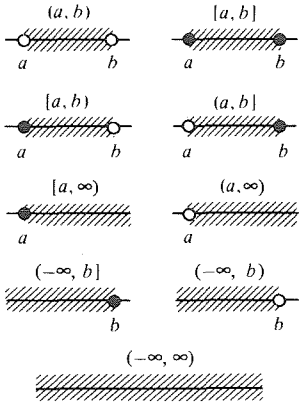

Figure R.2.1. The nine types of intervals.

We begin by listing the notations used for different kinds of intervals.

(a,b) means all x such that $a<x<b$ (open interval).
$[a,b]$ means all x such that $a\leqslant x\leqslant b$ (closed interval).
$[a,b)$ means all x such that $a\leqslant x<b$ (half-open interval).
$(a,b]$ means all x such that $a<x\leqslant b$ (half-open interval).
$[a,\infty)$ means all x such that $a\leqslant x$ (half-open interval).
(a,∞) means all x such that $a<x$ (open interval).
$(-\infty,b]$ means all x such that $x\leqslant b$ (half-open interval).
$(-\infty,b)$ means all x such that $x<b$ (open interval).
$(-\infty,\infty)$ means all real numbers (open interval).

These collections of real numbers are illustrated in Fig. R.2.1. A black dot indicates that the corresponding endpoint is included in the interval; a white circle indicates that the endpoint is not included in the interval. Notice that a closed interval contains both its endpoints, a half-open interval contains one endpoint, and an open interval contains none.

Warning The symbol ∞ ("infinity") does not denote a real number. It is merely a placeholder to indicate that an interval extends without limit.

In the formation of intervals, we can allow $a=b$. Thus the interval $[a,a]$ consists of the number a alone (if $a\leqslant x\leqslant a$, then $x=a$), while (a,a), $(a,a]$, and $[a,a)$ contain no numbers at all.

Many collections of real numbers are not intervals. For example, the integers, $\ldots,-3,-2,-1,0,1,2,3,\ldots$, form a collection of real numbers which cannot be designated as a single interval. The same goes for the rational numbers, as well as the collection of all x for which $x^2-2x-3>0$. (See Fig. R.1.3.) A collection of real numbers is also called a *set* of real numbers. Intervals are examples of sets of real numbers, but not every set is an interval.

We will often use capital letters to denote sets of numbers. If A is a set and x is a number, we write $x\in A$ and say that "x is an element of A" if x belongs to the collection A. For example, if we write $x\in[a,b]$ (read "x is an element of $[a,b]$"), we mean that x is a member of the collection $[a,b]$; that is, $a\leqslant x\leqslant b$. Similar notation is used for the other types of intervals.

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Example 1 True or false: (a) $3 \in [1,8]$, (b) $-1 \in (-\infty, 2)$, (c) $1 \in [0,1)$, (d) $8 \in (-8, \infty)$, (e) $3 - 5 \in \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of integers.

Solution (a) True, because $1 \leqslant 3 \leqslant 8$ is true;
(b) true, because $-1 < 2$ is true;
(c) false, because $0 \leqslant 1 < 1$ is false ($1 < 1$ is false);
(d) true, because $-8 < 8$ is true.
(e) true, because $3 - 5 = -2$, which is an integer. ▲

Example 2 Prove: If $a < b$, then $(a + b)/2 \in (a, b)$.

Solution We must show that $a < (a + b)/2 < b$. For the first inequality:

$$a < b,$$
$$2a < a + b \qquad \text{(adding } a\text{)},$$
$$a < \frac{a+b}{2} \qquad \text{(multiplying by } \tfrac{1}{2}\text{)}.$$

The proof that $(a + b)/2 < b$ is done similarly; add b to $(a < b)$ and divide by 2. Thus $a < (a + b)/2 < b$; i.e., the average of two numbers lies between them. ▲

Example 3 Let A be the set consisting of those x for which $x^2 - 2x - 3 > 0$. Describe A in terms of intervals.

Solution From Example 10(b), Section R.1, A consists of those x for which $x > 3$ or $x < -1$. In terms of intervals, A consists of $(3, \infty)$ and $(-\infty, -1)$, as in Fig. R.1.3. ▲

If a real number x is considered as a point on the number line, the distance between this point and zero is called the *absolute value* of x. If x is positive or zero, the absolute value of x is equal to x itself. If x is negative, the absolute value of x is equal to the positive number $-x$ (see Fig. R.2.2). The absolute value of x is denoted by $|x|$. For instance, $|8| = 8$, $|-7| = 7$, $|-10^8| = 10^8$.

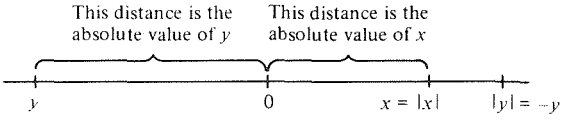

Figure R.2.2. The absolute value measures the distance to the origin.

Absolute Value

The absolute value $|x|$ of a real number x is equal to x if $x \geqslant 0$ and $-x$ if $x < 0$. To compute $|x|$, change the sign of x, if necessary, to make a non-negative number.

Example 4 Find (a) $|-6|$, (b) $|8 \cdot (-3)|$, (c) $|(-2) \cdot (-5)|$, (d) all x such that $|x| = 2$.

Solution (a) $|-6| = 6$.
(b) $|8 \cdot (-3)| = |-24| = 24$.
(c) $|(-2)(-5)| = |10| = 10$.
(d) If $|x| = 2$ and $x \geqslant 0$, we must have $x = 2$. If $|x| = 2$ and $x < 0$, we must have $-x = 2$; that is, $x = -2$. Thus $|x| = 2$ if and only if $x = \pm 2$. ▲

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For any real number x, $|x| \geqslant 0$, and $|x| = 0$ exactly when $x = 0$. If b is a positive number, there are two numbers having b as their absolute value: b and $-b$. Geometrically, if $x < 0$, $|x|$ is the "mirror image" point which is obtained from x by flipping the line over, keeping zero fixed.

If x_1 and x_2 are any two real numbers, the distance between x_1 and x_2 is $x_1 - x_2$ if $x_1 > x_2$ and $x_2 - x_1$ if $x_1 < x_2$. (See Fig. R.2.3 and note that the position of zero in this figure is unimportant.) Since $x_1 \geqslant x_2$ if and only if $x_1 - x_2 \geqslant 0$, and $x_2 - x_1 = -(x_1 - x_2)$, we have the result shown in the next box.

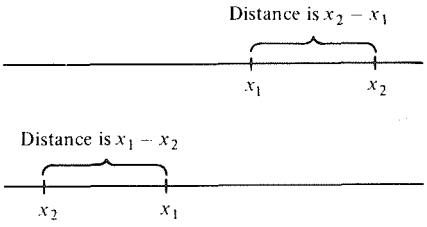

Figure R.2.3. The distance between x_1 and x_2 is $|x_1 - x_2|$.

Distance Formula on the Line

If x_1 and x_2 are points on the number line, the distance between x_1 and x_2 is equal to $|x_1 - x_2|$.

Example 5 Describe as an interval the set of real numbers x for which $|x - 8| \leqslant 3$.

Solution $|x - 8| \leqslant 3$ means that $x - 8 \leqslant 3$ in case $x - 8 \geqslant 0$ and $-(x - 8) \leqslant 3$ in case $x - 8 < 0$. In the first case, we have $x \leqslant 11$ and $x \geqslant 8$. In the second case, we have $x \geqslant 5$ and $x < 8$. Thus $|x - 8| \leqslant 3$ if and only if $x \in [5, 11]$. ▲

Example 6 Describe the interval $(4, 9)$ by a single inequality involving absolute values.

Solution Let m be the midpoint of the interval $(4, 9)$; that is, $m = \frac{1}{2}(4 + 9) = \frac{13}{2}$. A number x belongs to $(4, 9)$ if and only if the distance from x to m is less than the distance from 9 to m, which is $|9 - \frac{13}{2}| = \frac{5}{2}$. (Note that the distance from 4 to m is $|4 - \frac{13}{2}| = |-\frac{5}{2}| = \frac{5}{2}$ as well.) So we have $x \in (4, 9)$ if and only if $|x - \frac{13}{2}| < \frac{5}{2}$. (See Fig. R.2.4.) ▲

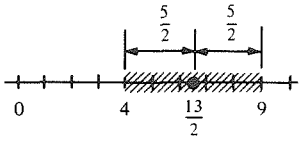

Figure R.2.4. The interval $(4, 9)$ may be described by the inequality $|x - \frac{13}{2}| < \frac{5}{2}$.

The most important algebraic properties of absolute values are listed below.

Properties of Absolute Values

If x and y are any real numbers:

1. $|x + y| \leqslant |x| + |y|$
2. $|xy| = |x|\,|y|$
3. $|x| = \sqrt{x^2}$

Example 7 Show by example that $|x + y|$ is not always equal to $|x| + |y|$.

Solution Let $x = 3$ and $y = -5$. Then $|x + y| = |3 - 5| = 2$, while $|x| + |y| = 3 + 5 = 8$. (Many other numbers will work as well. In fact, $|x + y|$ will be less than $|x| + |y|$ whenever x and y have opposite signs.) ▲

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The preceding example illustrates the general relation between $|x+y|$ and $|x|+|y|$: they are equal if x and y have the same sign, $|x+y|<|x|+|y|$ if x and y have opposite signs.

Example 8 Prove that $|x|=\sqrt{x^2}$.

Solution For any number x, we have $(-x)^2=x^2$, so $|x|^2=x^2$ whatever the sign of x. Thus $|x|$ is a number such that $|x|\geqslant 0$ and $|x|^2=x^2$, so it is the square root of x^2. ▲

Exercises for Section R.2

1. True or false: (a) $-7\in[-8,1]$; (b) $5\in(\frac{11}{2},6]$; (c) $4\in(-4,6]$; (d) $4\in(4,6)$; (e) $\frac{3}{2}+\frac{7}{2}\in\mathbb{Z}$?
2. Which numbers in the list $-54,-9,-\frac{2}{3},0,\frac{5}{16},\frac{1}{2}$, $8,32,100$ belong to which of the following intervals?
   (a) $[-10,1)$ (b) $(-\infty,44)$
   (c) $(75,500)$ (d) $(-20,-\frac{2}{3})$
   (e) $(-9,\frac{5}{16}]$
3. Prove: If $a<b$ then $a<(2a+b)/3$.
4. Prove: If $a<b$ then $(a-4b)/3<-b$.

Describe the solutions of the inequalities in Exercises 5–12 in terms of intervals.

5. $x+4\geqslant 7$
6. $2x+5\leqslant -x+1$
7. $x>4x-6$
8. $5-x>4-2x$
9. $x^2+2x-3>0$
10. $2x^2-6\leqslant 0$
11. $x^2-x\geqslant 0$
12. $(2x+1)(x-5)\leqslant 0$

Find the absolute values in Exercises 13–20.

13. $|3-5|$
14. $|3+5|$
15. $|-3-5|$
16. $|-3+5|$
17. $|3\cdot 5|$
18. $|(-3)(-5)|$
19. $|(-3)\cdot 5|$
20. $|3\cdot(-5)|$
21. Find all x such that $|x|=8$.
22. Find all x such that $|-x|=9$.

Describe in terms of absolute values the set of x satisfying the inequalities in Exercises 23–26.

23. $x^2+5x>0$
24. $x^2-2x<0$
25. $x^2-x-2>0$
26. $x^2+5x+7<0$

Express each of the inequalities in Exercises 27–32 in the form "x belongs to the interval ...".

27. $3<x\leqslant 4$
28. $x>5$
29. $|x|<5$
30. $|x-3|\leqslant 6$
31. $|3x+1|<2$
32. $x^2-3x+2\geqslant 0$

Express each of the statements in Exercises 33–38 in terms of an inequality involving absolute values:

33. $x\in(-3,3)$
34. $-x\in(-4,4)$
35. $x\in(-6,6)$
36. $x\in(2,6)$
37. $x\in[-8,12]$
38. $|x|\in(0,1)$
39. Show by example that $|x+y+z|$ is not always equal to $|x|+|y|+|z|$.
40. Show by example that $|x-y|$ need not equal $||x|-|y||$.
41. Prove that $x=\sqrt[3]{x^3}$.
42. Prove that $|x|=\sqrt[4]{x^4}$.
43. Is the formula $|x-y|\leqslant|x|-|y|$ always true?
44. Using the formula $|xy|=|x||y|$, find a formula for $|a/b|$. [*Hint*: Let $x=b$ and $y=a/b$.]

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R.3 Laws of Exponents

Fractional and integer exponents obey similar laws.

The expression b^n, where b is a real number called the *base* and n is a natural number called the *exponent*, is defined as the product of b with itself n times:

$$b^n = b \cdot b \cdot \cdots \cdot b \qquad (n \text{ times}).$$

This operation of raising a number to a power, or *exponentiation*, has the following properties, called *laws of exponents*:

Laws of Exponents: Integer Powers

1. $b^n b^m = b^{n+m}$
2. $(b^n)^m = b^{nm}$
3. $(bc)^n = b^n c^n$

These laws can all be understood and remembered using common sense. For example, $b^{m+n} = b^m b^n$ because

$$\underbrace{b \cdot b \cdot b \cdot \cdots \cdot b}_{m+n \text{ times}} = \underbrace{\underbrace{(b \cdot b \cdot \cdots \cdot b)}_{m \text{ times}} \underbrace{(b \cdot b \cdot \cdots \cdot b)}_{n \text{ times}}}_{m+n \text{ times in all}}.$$

Likewise

$$\begin{aligned}(b^n)^m &= \underbrace{(b^n)(b^n)\cdots(b^n)}_{m \text{ times}} \\ &= \underbrace{(b \cdots b)(b \ldots b)(b \ldots b)}_{nm \text{ times}} \\ &= b^{nm}.\end{aligned}$$

and

$$\begin{aligned}(bc)^n &= \underbrace{bcbc \ldots bc}_{n \text{ times}} = \underbrace{(b \ldots b)}_{n \text{ times}} \underbrace{(c \ldots c)}_{n \text{ times}} \\ &= b^n c^n.\end{aligned}$$

Example 1 Simplify

$$\text{(a)} \quad 2^{10} \cdot 5^{10} \qquad \text{(b)} \quad \frac{(3 \cdot 2)^{10} + 3^9}{3^9}$$

Solution (a) $2^{10} \cdot 5^{10} = (2 \cdot 5)^{10} = 10^{10}$.

(b) $$\frac{(3 \cdot 2)^{10} + 3^9}{3^9} = \frac{3^{10} \cdot 2^{10} + 3^9}{3^9} = \frac{3^9 \cdot 3 \cdot 2^{10} + 3^9}{3^9} = 3 \cdot 2^{10} + 1.$$ ▲

The first of the three laws of exponents is particularly important; it is the basis for extending the operation of exponentiation to allow negative exponents. If b^0 were defined, we ought to have $b^0 b^n = b^{0+n} = b^n$. If $b \neq 0$, then $b^n \neq 0$, and the equation $b^0 b^n = b^n$ implies that b^0 must be 1. We take this as the *definition* of b^0, noting that 0^0 is not defined (see Exercise 31).

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If n is a natural number, then b^{-n} is defined in order to make $b^{-n}b^n = b^{-n+n} = b^0 = 1$; that is, $b^{-n} = 1/b^n$.

Negative Powers: Definition and Laws of Exponents

If b is a real number and n is a positive integer, we define

$$b^{-n} = \frac{1}{b^n}.$$

The laws of exponents given in the preceding box remain valid for integers n, m; positive, negative, or zero.

For example, let us show that $b^{n+m} = b^n b^m$ is valid if $n = -q$ is negative and m is positive, with $m > q$. Then $b^{n+m} = b^{m-q} = b \cdot \cdots \cdot b$ ($m - q$ times). Also,

$$b^n b^m = b^{-q} b^m = \frac{b^m}{b^q} = \frac{\overbrace{b \cdots bb}^{m \text{ times}}}{\underbrace{b \ldots b}_{q \text{ times}}} = \overbrace{b \ldots b}^{m-q \text{ times}}.$$

Thus $b^{n+m} = b^n b^m$. The other cases and laws are verified similarly.

Example 2 Simplify

(a) $\dfrac{(2 \cdot 3)^{-2} \cdot 4}{(1/3)^2}$.

(b) $[(8/3)^2 - (3/8)^3][(8/3)^{-2} + (3/8)^{-3}]$.

Solution (a) $\dfrac{(2 \cdot 3)^{-2} 4}{(1/3)^2} = \dfrac{4 \cdot 3^2}{2^2 \cdot 3^2} = 1.$

(b) Multiplying out, the given expression becomes

$$\left(\frac{8}{3}\right)^2\left(\frac{8}{3}\right)^{-2} + \left(\frac{8}{3}\right)^2\left(\frac{3}{8}\right)^{-3} - \left(\frac{3}{8}\right)^3\left(\frac{8}{3}\right)^{-2} - \left(\frac{3}{8}\right)^3\left(\frac{3}{8}\right)^{-3}$$

$$= 1 + \left(\frac{8}{3}\right)^2\left(\frac{8}{3}\right)^3 - \left(\frac{3}{8}\right)^3\left(\frac{3}{8}\right)^2 - 1$$

$$= \left(\frac{8}{3}\right)^5 - \left(\frac{3}{8}\right)^5. \blacktriangle$$

To define $b^{1/n}$, we require $b^{1/n} \cdot b^{1/n} \cdot \ldots \cdot b^{1/n} = b^{1/n + \cdots + 1/n} = b^1 = b$ to hold—that is, $(b^{1/n})^n$ ought to be equal to b. Thus we declare $b^{1/n}$ to be $\sqrt[n]{b}$, that positive number whose nth power is b; i.e., $b^{1/n}$ is defined by the equation $(b^{1/n})^n = b$. (If n is odd, then $\sqrt[n]{b}$ may be defined even if b is negative, but we will reserve the notation $b^{1/n}$ for the case $b > 0$.)

Finally, if $r = m/n$ is a rational number, we define $b^r = b^{m/n} = (b^m)^{1/n}$. We leave it to you to verify that the result is independent of the way in which r is expressed as a quotient of positive integers; for instance, $(b^4)^{1/6} = (b^6)^{1/9}$ (see Exercise 32).

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Having defined b^r for $b > 0$ and r rational, one can go back and prove the laws of exponents for this general case. These laws are useful for calculations with rational exponents.

Let us first check that $(bc)^{1/n} = b^{1/n}c^{1/n}$. Now $(bc)^{1/n}$ is that number whose nth power is bc; but $(b^{1/n}c^{1/n})^n = (b^{1/n})^n(c^{1/n})^n = bc$, by Property 3 for integer powers and the fact that $(b^{1/n})^n = b$. Thus $(b^{1/n}c^{1/n})^n = bc$, which means that $b^{1/n}c^{1/n} = (bc)^{1/n}$.

Using this, we can check that $b^{p+q} = b^p b^q$ as follows. Let $p = m/n$ and $q = k/l$. Then

$$
\begin{aligned}
b^{p+q} = b^{m/n+k/l} &= b^{(ml+kn)/nl} = \left(b^{ml+kn}\right)^{1/nl} \\
&= \left(b^{ml}b^{kn}\right)^{1/nl} && \text{(by Property 1 for integer powers)} \\
&= \left(b^{ml}\right)^{1/nl}\left(b^{kn}\right)^{1/nl} && \left(\text{by the law } (bc)^{1/n} = b^{1/n}c^{1/n} \text{ just proved}\right) \\
&= b^{ml/nl}b^{kn/nl} && \left(\text{by the definition of } b^{m/n}\right) \\
&= b^p b^q.
\end{aligned}
$$

The other properties are checked in a similar way (Exercises 33 and 34).

Rational Powers

Rational powers are defined by:

$b^n = b \cdot \cdots \cdot b$ (n times); $b^0 = 1$
$b^{-n} = 1/b^n$
$b^{1/n} = \sqrt[n]{b}$ if $b > 0$ and n is a natural number
$b^{m/n} = (b^m)^{1/n}$

If $b, c > 0$ and p, q are rational, then:

1. $b^{p+q} = b^p b^q$
2. $b^{pq} = (b^p)^q$
3. $(bc)^p = b^p c^p$
4. $b^p < b^q$ if $b > 1$ and $p < q$; $b^p > b^q$ if $b < 1$ and $p < q$.

Example 3 Find $8^{-2/3}$ and $9^{3/2}$.

Solution $8^{-2/3} = 1/8^{2/3} = 1/(\sqrt[3]{8})^2 = 1/2^2 = 1/4$. $\quad 9^{3/2} = (\sqrt{9})^3 = 3^3 = 27$. ▲

Example 4 Simplify $[x^{2/3}(x^{-3/2})]^{8/3}$ and $(x^{2/3})^{5/2}/x^{1/4}$.

Solution $(x^{2/3}x^{-3/2})^{8/3} = (x^{2/3-3/2})^{8/3} = (x^{-5/6})^{8/3} = x^{-20/9} = 1/\sqrt[9]{x^{20}}$.

$(x^{2/3})^{5/2}/x^{1/4} = x^{2/3\cdot 5/2-1/4} = x^{5/3-1/4} = x^{17/12}$. ▲

Example 5 We defined $b^{m/n}$ as $(b^m)^{1/n}$. Show that $b^{m/n} = (b^{1/n})^m$ as well.

Solution We must show that $(b^{1/n})^m$ is the nth root of b^m. But $[(b^{1/n})^m]^n = (b^{1/n})^{mn} = (b^{1/n})^{nm} = [(b^{1/n})^n]^m = b^m$; this calculation used only the laws of *integer* exponents and the fact that $(b^{1/n})^n = b$. ▲

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Example 6 Remove the square roots in the denominator:

$$\frac{1}{\sqrt{x-a}+\sqrt{x-b}}.$$

Solution There is a useful trick called *rationalizing*. We multiply top and bottom by $\sqrt{x-a}-\sqrt{x-b}$, giving

$$\frac{\sqrt{x-a}-\sqrt{x-b}}{(\sqrt{x-a}+\sqrt{x-b})(\sqrt{x-a}-\sqrt{x-b})} = \frac{\sqrt{x-a}-\sqrt{x-b}}{(\sqrt{x-a})^2-(\sqrt{x-b})^2}$$

$$= \frac{\sqrt{x-a}-\sqrt{x-b}}{b-a}. \blacktriangle$$

Example 7 Assume that the cost of food doubles every 6 years. By what factor has it increased after

(a) 12 years? (b) 18 years? (c) 3 years? (d) 20 years?

Solution (a) Since the cost doubles in 6 years, in 12 years it increases by a factor of $2 \cdot 2 = 4$.

(b) In $18 = 6 + 6 + 6$ years it increases by a factor $2 \cdot 2 \cdot 2 = 8$.

(c) In 3 years, let it increase by a factor k. Then in 6 years we get $k \cdot k = 2$, so $k = \sqrt{2} = 2^{3/6} \approx 1.4142$.

(d) In 20 years the factor is $2^{20/6} = 2^{10/3} = \sqrt[3]{2^{10}} \approx 10.0794$. $\blacktriangle$

Exercises for Section R.3

Simplify the expressions in Exercises 1–20.

1. $3^2 \cdot \left(\frac{1}{3}\right)^2$
2. $8^3 \cdot \left(\frac{1}{4}\right)^2$
3. $\dfrac{(4 \cdot 3)^{10} + 4^9}{8^4}$
4. $\dfrac{(2 \cdot 3)^{16} + 3^{15}}{3^{15}}$
5. $\dfrac{(4 \cdot 3)^{-6} \cdot 8}{9^3}$
6. $\dfrac{(27)^{-1}}{3^{-3}}$
7. $\left(\dfrac{8^{-2}}{4^{-4}}\right)^4$
8. $\left[\left(\frac{1}{2}\right)^{-1} + \left(\frac{1}{3}\right)^{-3}\left(\frac{1}{2}\right)^{-1} - \left(\frac{1}{3}\right)^{-2}\right]$
9. $9^{1/2}$
10. $16^{1/4}$
11. $(1/9)^{-1/2}$
12. $(1/16)^{-1/4}$
13. $2^{5/3}/4^{7/3}$
14. $3^{-8/11}(1/9)^{-4/11}$
15. $12^{2/3} \cdot 18^{2/3}$
16. $20^{7/2} \cdot 5^{-7/2}$
17. $(x^{3/2} + x^{5/2})x^{-3/2}$
18. $(x^{3/4})^{8/3}$
19. $x^{5/2}(x^{-3/2} + 2x^{1/2} + 3x^{7/2})$
20. $y^{1/2}(1/y + 2\sqrt{y} + y^{-1/3})$

Using the laws of rational exponents, verify the root formulas in Exercises 21 and 22.

21. $\sqrt[a]{\sqrt[b]{x}} = \sqrt[ab]{x}$
22. $\sqrt[ac]{x^{ab}} = \sqrt[c]{x^b}$

Simplify the expressions in Exercises 23 and 24 by writing with rational exponents.

23. $\left[\dfrac{\sqrt[4]{ab^3}}{\sqrt{b}}\right]^6$
24. $\sqrt[3]{\dfrac{\sqrt{a^3b^9}}{\sqrt[4]{a^6b^6}}}$

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25. The price of housing doubles every 10 years. By what factor does it increase after 20 years? 30 years? 50 years?

[1]26. Money in a certain bank account grows by a factor of 1.1 every year. If an initial deposit of \$100 is made, how much money will be in the account after 10 years?

Factor the expressions in Exercises 27–30 using fractional exponents. For example: $x + 2\sqrt{2xy} + 2y = (x^{1/2} + (2y)^{1/2})^2$.

27. $x - \sqrt{xy} - 2y$

28. $\sqrt[3]{xy^2} + \sqrt[3]{yx^2} + x + y$

29. $x - 2\sqrt{x} - 8$

30. $x + 2\sqrt{3x} + 3$

★31. Since $0^x = 0$ for any positive rational x, 0^0 ought to be zero. On the other hand, $b^0 = 1$ for any $b > 0$, so 0^0 ought to be 1. Are *both* choices consistent with the laws of exponents?

★32. Suppose that $b > 0$ and that $p = m/n = m'/n'$. Show, using the definition of rational powers, that $b^{m/n} = b^{m'/n'}$; that is, b^p is unambiguously defined. [*Hint:* Raise both $b^{m/n}$ and $b^{m'/n'}$ to the power nn'.]

★33. Prove Rules 2 and 3 for rational powers.

★34. Let $b > 1$ and p and q be rational numbers with $p < q$. Prove that $b^p < b^q$. Deduce the corresponding result for $b < 1$ by using $b^p = (1/b)^{-p}$.

R.4 Straight Lines

The graph $y = ax + b$ is a straight line in the xy plane.

In this section, we review some basic analytic geometry. We will develop the point-slope form of the equation of a straight line which will be essential for calculus.

One begins the algebraic representation of the plane by drawing two perpendicular lines, called the x and y axes, and the placing the real numbers on each of these lines, as shown in Fig. R.4.1. Any point P in the plane can

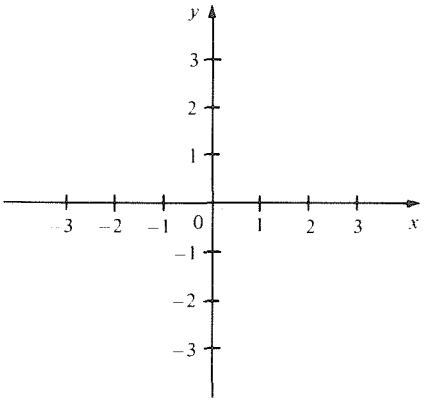

Figure R.4.1. The x and y axes in the plane.

now be described by the pair (a, b) of real numbers obtained by dropping perpendiculars to the x and y axes, as shown in Fig. R.4.2. The numbers which describe the point P are called the *coordinates* of P: the first coordinate listed is called the x coordinate; the second is the y coordinate. We can use any letters we wish for the coordinates, including x and y themselves.

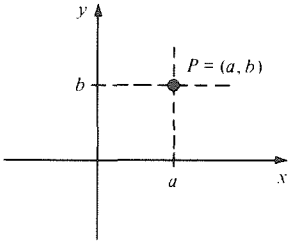

Figure R.4.2. The point P has coordinates (a, b).

Often the point with coordinates (a, b) is simply called "the point (a, b)." Drawing a point (a, b) on a graph is called *plotting* the point; some points are plotted in Fig. R.4.3. Note that the point $(0, 0)$ is located at the intersection of the coordinate axes; it is called the *origin* of the coordinate system.

[1] This symbol denotes exercises or discussions that may require use of a hand-held calculator.

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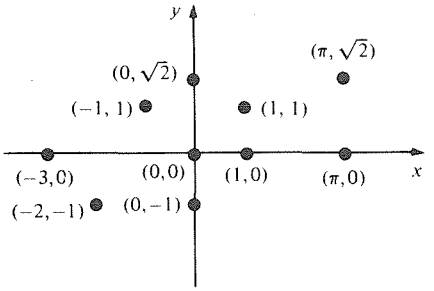

Figure R.4.3. Examples of plotted points.

Example 1 Let $a = 3$ and $b = 2$. Plot the points (a, b), (b, a), $(-a, b)$, $(a, -b)$, and $(-a, -b)$.

Solution The points to be plotted are $(3, 2)$, $(2, 3)$, $(-3, 2)$, $(3, -2)$, and $(-3, -2)$; they are shown in Fig. R.4.4. ▲

The theorem of Pythagoras leads to a simple formula for the distance between two points (see Fig. R.4.5):

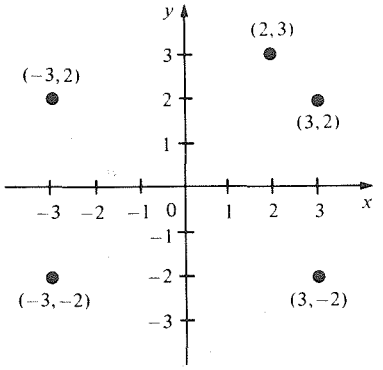

Figure R.4.4. More plotted points.

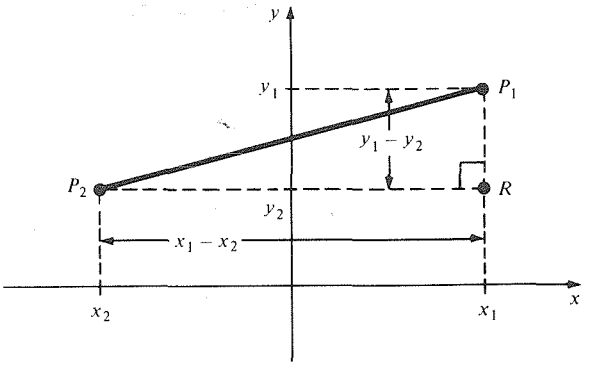

Figure R.4.5. By the Pythagorean theorem, $|P_1P_2| = \sqrt{|P_2R|^2 + |P_1R|^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Distance Formula

If P_1 has coordinates (x_1, y_1) and P_2 has coordinates (x_2, y_2), the distance from P_1 to P_2 is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. The distance between P_1 and P_2 is denoted $|P_1P_2|$.

Example 2 Find the distance from $(6, -10)$ to $(2, -1)$.

Solution The distance is

$$\sqrt{(6-2)^2 + [-10-(-1)]^2} = \sqrt{4^2 + (-9)^2}$$
$$= \sqrt{16 + 81} = \sqrt{97} \approx 9.85. \blacktriangle$$

If we have two points on the x axis, $(x_1, 0)$ and $(x_2, 0)$, the distance between them is $\sqrt{(x_1 - x_2)^2 + (0-0)^2} = \sqrt{(x_1 - x_2)^2} = |x_1 - x_2|$. Thus *the distance formula in the plane includes the distance formula on the line* as a special case.

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Draw a line[2] l in the plane and pick two distinct points P_1 and P_2 on l. Let P_1 have coordinates (x_1, y_1) and P_2 have coordinates (x_2, y_2). The ratio $(y_2 - y_1)/(x_2 - x_1)$ (assuming that $x_2 \neq x_1$) is called the *slope* of the line l and is often denoted by the letter m. See Fig. R.4.6.

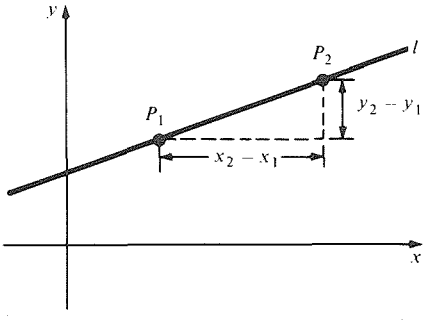

Figure R.4.6. The slope of this line is $(y_2 - y_1)/(x_2 - x_1)$.

Slope Formula

If (x_1, y_1) and (x_2, y_2) lie on the line l, the slope of l is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

An important feature of the slope m is that it does not depend upon which two points we pick, so long as they lie on the line l. To verify this, we observe (see Fig. R.4.7) that the right triangles P_1P_2R and $P_1'P_2'R'$ are similar, since corresponding angles are equal, so $P_2R/P_1R = P_2'R'/P_1'R'$. In other words, the slope calculated using P_1 and P_2 is the same as the slope calculated using P_1' and P_2'. The slopes of some lines through the origin are shown in Fig. R.4.8.

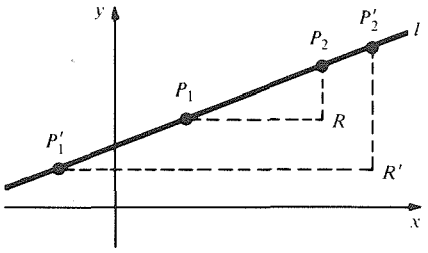

Figure R.4.7. The slope does not depend on which two points on l are used.

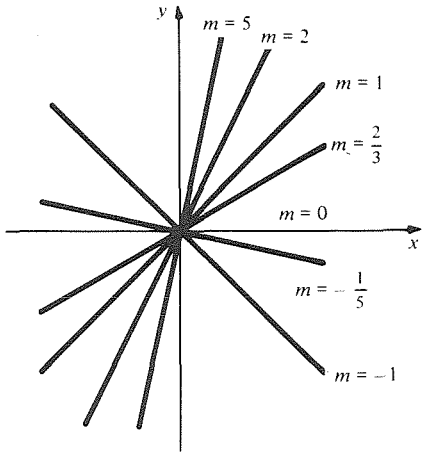

Figure R.4.8. Slopes of some lines through the origin.

Example 3 What is the slope of the line which passes through the points (0, 1) and (1, 0)?

Solution By the slope formula, with $x_1 = 0$, $y_1 = 1$, $x_2 = 1$, and $y_2 = 0$, the slope is $(0 - 1)/(1 - 0) = -1$. ▲

[2] In this book when we use the term *line* with no other qualification, we shall mean a *straight line*. Rather than referring to "curved lines," we will use the term *curve*.

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Warning A line which is parallel to the y axis does not have a slope. In fact, any two points on such a line have the same x coordinates, so when we form the ratio $(y_2 - y_1)/(x_2 - x_1)$, the denominator becomes zero, which makes the expression meaningless. A vertical line has the equation $x = x_1$; y can take any value.

To find the equation satisfied by the coordinates of the points on a line, we consider a line l with slope m and which passes through the point (x_1, y_1). If (x, y) is any *other* point on l, the slope formula gives

$$\frac{y - y_1}{x - x_1} = m.$$

That is,

$$y = y_1 + m(x - x_1).$$

This is called the *point-slope form* of the equation of l; a general point (x, y) lies on l exactly when the equation holds.

If, for the point in the point-slope form of the equation, we take the point $(0, b)$ where l intersects the y axis (the number b is called the y *intercept* of l), we have $x_1 = 0$ and $y_1 = b$ and obtain the *slope-intercept form* $y = mx + b$.

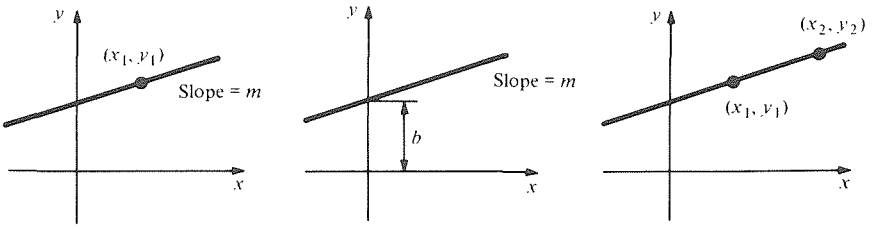
If we are given two points (x_1, y_1) and (x_2, y_2) on a line, we know that the slope is $(y_2 - y_1)/(x_2 - x_1)$. Substituting this term for m in the point-slope form of the equation gives the *point-point form*:

$$y = y_1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1).$$

Straight Lines

Name	*Data needed*	*Formula*
point-slope	one point (x_1, y_1) on the line and the slope m	$y = y_1 + m(x - x_1)$
slope-intercept	the slope m of the line and the y-intercept b	$y = mx + b$
point-point	two points (x_1, y_1) and (x_2, y_2) on the line	$y = y_1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$

For calculus, the point-slope form will turn out to be the most important of the three forms of the equation of a line, illustrated in Fig. R.4.9(a).



(a) Point-Slope: $y = y_1 + m(x - x_1)$

(b) Slope-Intercept: $y = mx + b$

(c) Point-Point: $y = y_1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$

Figure R.4.9. Three forms for the equation of a line.

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Example 4 Find the equation of the line through (1, 1) with slope 5. Put the equation into slope-intercept form.

Solution Using the point-slope form, with $x_1 = 1$, $y_1 = 1$, and $m = 5$, we get $y = 1 + 5(x - 1)$. This simplifies to $y = 5x - 4$, which is the slope-intercept form. ▲

Example 5 Let l be the line through the points (3, 2) and (4, −1). Find the point where this line intersects the x axis.

Solution The equation of the line, in point-point form, with $x_1 = 3$, $y_1 = 2$, $x_2 = 4$, and $y_2 = -1$, is

$$y = 2 + \left(\frac{-1-2}{4-3}\right)(x-3)$$
$$= 2 - 3(x-3).$$

The line intersects the x axis at the point where $y = 0$, that is, where

$$0 = 2 - 3(x-3).$$

Solving this equation for x, we get $x = \frac{11}{3}$, so the point of intersection is $(\frac{11}{3}, 0)$. (See Fig. R.4.10.) ▲

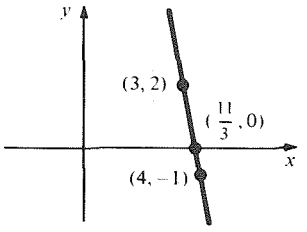

Figure R.4.10. Finding where the line through (3, 2) and (4, −1) meets the x axis.

Example 6 Find the slope and y intercept of the line $3y + 8x + 5 = 0$.

Solution The following equations are equivalent:

$$3y + 8x + 5 = 0,$$
$$3y = -8x - 5,$$
$$y = -\tfrac{8}{3}x - \tfrac{5}{3}.$$

The last equation is in slope-intercept form, with slope $-\frac{8}{3}$ and y intercept $-\frac{5}{3}$. ▲

Using the method of Example 6, one can show that any equation of the form $Ax + By + C = 0$ describes a straight line, as long as A and B are not both zero. If $B \neq 0$, the slope of the line is $-A/B$; if $B = 0$, the line is vertical; and if $A = 0$, it is horizontal.

Finally, we recall without proof the fact that lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$. In other words, *the slopes of perpendicular lines are negative reciprocals of each other.*

Example 7 Find the equation of the line through (0, 0) which is perpendicular to the line $3y - 2x + 8 = 0$.

Solution The given equation has the form $Ax + By + C = 0$, with $A = -2$, $B = 3$, and $C = 8$; the slope of the line it describes is $-A/B = \frac{2}{3} = m_1$. The slope m_2 of the perpendicular line must satisfy $m_1 m_2 = -1$, so $m_2 = -\frac{3}{2}$. The line through the origin with this slope has the equation $y = -\frac{3}{2}x$. ▲

Exercises for Section R.4

1. Plot the points (0, 0), (1, 1), (−1, −1), (2, 8), (−2, −8), (3, 27), and (−3, −27).
2. Plot the points (−1, 2), (−1, −2), (1, −2), and (1, 2).
3. Plot the points (x, x^2) for $x = -2, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2$.
4. Plot the points $(x, x^4 - x^2)$ for $x = -2, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2$.

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Find the distance between each of the pairs of points in Exercises 5–10.

5. $(1, 1), (1, -1)$
6. $(-1, 1), (-1, -1)$
7. $(-3, 9), (2, -8)$
8. $(0, 0), (-3, 27)$
9. $(43721, 56841), (3, 56841)$
10. $(839, 8400), (840, 8399)$

Find the distance or a formula for the distance between each pair of points in Exercises 11–16.

11. $(2, 1), (3, 2)$
12. $(a, 2), (3 + a, 6)$
13. $(x, y), (3x, y + 10)$
14. $(a, 0), (a + b, b)$
15. $(a, a), (-a, -a)$
16. $(a, b), (10a, 10b)$

Find the slope of the line through the points in Exercises 17–20.

17. $(1, 3), (2.6)$
18. $(0, 1), (2, -4)$
19. $(-1, 6), (1, -1)$
20. $(0, 0), (-1, -1)$

In Exercises 21–24 find the equation of the line through the point P with slope m, and sketch a graph of the line.

21. $P = (2, 3)$, $m = 2$
22. $P = (-2, 6)$, $m = -\frac{1}{2}$
23. $P = (-1, 7)$, $m = 0$
24. $P = (7, -1)$, $m = 1$

Find the equation of the line through the pairs of points in Exercises 25–28.

25. $(5, 7), (-1, 4)$
26. $(1, 1), (3, 2)$
27. $(1, 4), (3, 4)$
28. $(1, 4), (1, 6)$

Find the slope and y intercept of each of the lines in Exercises 29–36.

29. $x + 2y + 4 = 0$
30. $\frac{1}{2}x - 3y + \frac{1}{3} = 0$
31. $4y = 17$
32. $2x + y = 0$
33. $13 - 4x = 7(x + y)$
34. $x - y = 14(x + 2y)$
35. $y = 17$
36. $x = 60$

37. (a) Find the slope of the line $4x + 5y - 9 = 0$.
    (b) Find the equation of the line through $(1, 1)$ which is perpendicular to the line in part (a).
38. (a) Find the slope of the line $2x - 8y - 10 = 0$.
    (b) Find the equation of the line through $(1, 0)$ which is perpendicular to the line in (a).

Find the equation of the line with the given data in Exercises 39–42.

39. Slope = 5; y intercept = 14
40. y intercept = 6; passes through $(7, 8)$
41. Passes through $(4, 2)$ and $(2, 4)$
42. Passes through $(-1, -1)$; slope $= -10$

43. Find the coordinates of the point which is a distance 3 from the x axis and a distance 5 from $(1, 2)$.

★44. (a) Find the coordinates of a point whose distance from $(0, 0)$ is $2\sqrt{2}$ and whose distance from $(4, 4)$ is $2\sqrt{2}$.
    (b) If $\lambda > 2\sqrt{2}$ show both algebraically and geometrically that there are exactly two points whose distance from $(0, 0)$ is λ and whose distance from $(4, 4)$ is also λ.

R.5 Circles and Parabolas

$(x - a)^2 + (y - b)^2 = r^2$ is a circle and $y = a(x - p)^2 + q$ is a parabola.

We now consider two more geometric figures which can be described by simple algebraic formulas: the circle and the parabola.

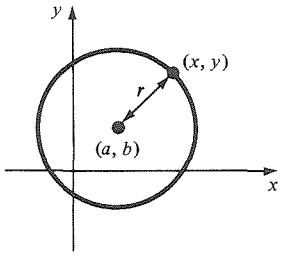

Figure R.5.1. The point (x, y) is a typical point on the circle with radius r and center (a, b).

The circle C with radius $r > 0$ and center at (a, b) consists of those points (x, y) for which the distance from (x, y) to (a, b) is equal to r. (See Fig. R.5.1.) The distance formula shows that $\sqrt{(x - a)^2 + (y - b)^2} = r$ or, equivalently, $(x - a)^2 + (y - b)^2 = r^2$. If the center of the circle is at the origin, this equation takes the simpler form $x^2 + y^2 = r^2$.

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Example 1 Find the equation of the circle with center $(1,0)$ and radius 5.

Solution Here $a = 1$, $b = 0$, and $r = 5$, so $(x-a)^2 + (y-b)^2 = r^2$ becomes the equation $(x-1)^2 + y^2 = 25$ or $x^2 - 2x + y^2 = 24$. ▲

Example 2 Find the equation of the circle whose center is $(2, 1)$ and which passes through the point $(5,6)$.

Solution The equation must be of the form $(x-2)^2 + (y-1)^2 = r^2$; the problem is to determine r^2. Since the point $(5,6)$ lies on the circle, it must satisfy the equation. That is,

$$r^2 = (5-2)^2 + (6-1)^2 = 3^2 + 5^2 = 34,$$

so the correct equation is $(x-2)^2 + (y-1)^2 = 34$. ▲

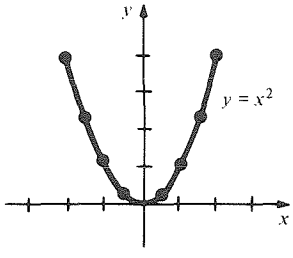
Example 3 Show that the graph of $x^2 + y^2 - 6x - 16y + 8 = 0$ is a circle. Find its center.

Solution Complete the squares:

$$\begin{aligned} 0 &= x^2 + y^2 - 6x - 16y + 8 = x^2 - 6x + y^2 - 16y + 8 \\ &= (x^2 - 6x + 9) + (y^2 - 16y + 64) - 9 - 64 + 8 \\ &= (x-3)^2 + (y-8)^2 - 65. \end{aligned}$$

Thus the equation becomes $(x-3)^2 + (y-8)^2 = 65$, whose graph is a circle with center $(3,8)$ and radius $\sqrt{65} \approx 8.06$. ▲

Consider next the equation $y = x^2$. If we plot a number of points whose coordinates satisfy this equation, by choosing values for x and computing y, we find that these points may be joined by a smooth curve as in Fig. R.5.2. This curve is called a *parabola*. It is also possible to give a purely geometric definition of a parabola and derive the equation from geometry as was done for the line and circle. In fact, we will do so in Section 14.1.



x	2	1.5	1	0.5	0
y	4	2.25	1	0.25	0

Figure R.5.2. The parabola $y = x^2$.

If x is replaced by $-x$, the value of y is unchanged, so the graph is symmetric about the y axis. Similarly, we can plot $y = 3x^2$, $y = 10x^2$, $y = -\frac{1}{2}x^2$, $y = -8x^2$, and so on. (See Fig. R.5.3.) These graphs are also parabolas. The general parabola of this type has the equation $y = ax^2$, where a

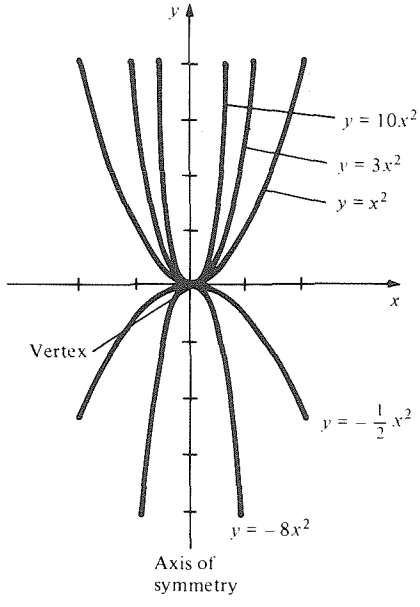


Figure R.5.3. Parabolas $y = ax^2$ for various values of a.

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is a nonzero constant; these parabolas all have their *vertex* at the origin. If $a > 0$ the parabola opens upwards, and if $a < 0$ it opens downwards.

Example 4 Let C be the parabola with vertex at the origin and passing through the point $(2, 8)$. Find the point on C whose x coordinate is 10.

Solution The equation of C is of the form $y = ax^2$. To find a, we use the fact that $(2, 8)$ lies on C. Thus $8 = a \cdot 2^2 = 4a$, so $a = 2$ and the equation is $y = 2x^2$. If the x coordinate of a point on C is 10, the y coordinate is $2 \cdot 10^2 = 200$, so the point is $(10, 200)$. ▲

A special focusing property of parabolas is of practical interest: a parallel beam of light rays (as from a star) impinging upon a parabola in the direction of its axis of symmetry will focus at a single point as shown in Fig. R.5.4. The property follows from the law that the angle of incidence equals the angle of reflection, together with some geometry or calculus. (See Review Exercises 86 and 87 at the end of Chapter 1.)

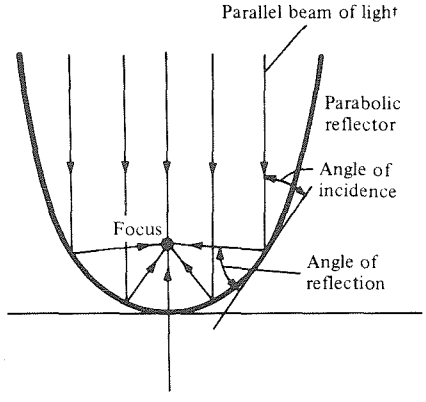

Figure R.5.4. The focusing property of a parabolic reflector.

Just as we considered circles with center at an arbitrary point (a, b), we can consider parabolas with vertex at any point (p, q). The equation of such a figure is $y = a(x - p)^2 + q$. We have started with $y = ax^2$, then replaced x by $x - p$ and y by $y - q$ to get

$$y - q = a(x - p)^2, \qquad \text{i.e.,} \quad y = a(x - p)^2 + q.$$

This process is illustrated in Fig. R.5.5. Notice that if (x, y) lies on the (shifted) parabola, then the corresponding point on the original parabola is $(x - p, y - q)$, which must therefore satisfy the equation of the original parabola; i.e., $y - q$ must equal $a(x - p)^2$.

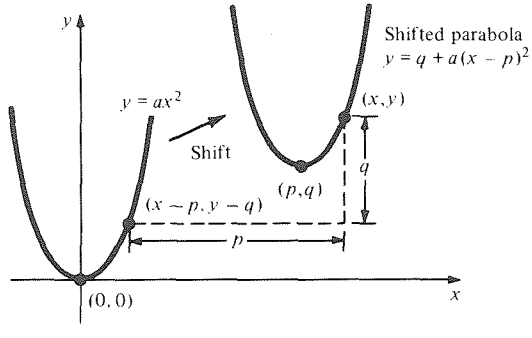

Figure R.5.5. The equation $y = q + a(x - p)^2$ is the parabola $y = ax^2$ shifted from $(0, 0)$ to (p, q).

Given an equation of the form $y = ax^2 + bx + c$, we can complete the square on the right-hand side to put it in the form $y = a(x - p)^2 + q$. Thus the graph of any equation $y = ax^2 + bx + c$ is a parabola.

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Example 5 Graph $y = -2x^2 + 4x + 1$.

Solution Completing the square gives

$$y = -2x^2 + 4x + 1 = -2\left(x^2 - 2x - \tfrac{1}{2}\right)$$
$$= -2\left(x^2 - 2x + 1 - 1 - \tfrac{1}{2}\right) = -2\left(x^2 - 2x + 1 - \tfrac{3}{2}\right)$$
$$= -2(x-1)^2 + 3.$$

The vertex is thus at $(1,3)$ and the parabola opens downward like $y = -2x^2$ (see Fig. R.5.6). ▲

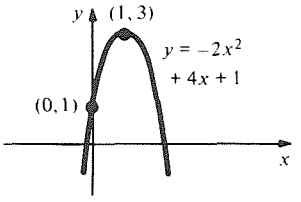

Figure R.5.6. A parabola with vertex at $(1,3)$.

Equations of Circles and Parabolas

The equation of the circle with radius r and center at (a,b) is

$$(x-a)^2 + (y-b)^2 = r^2.$$

The equation of a parabola with vertex at (p,q) is

$$y = a(x-p)^2 + q.$$

Analytic geometry provides an algebraic technique for finding the points where two geometric figures intersect. If each figure is given by an equation in x and y, we solve for those pairs (x, y) which satisfy both equations.

Two lines will have either zero, one, or infinitely many intersection points; there are none if the two lines are parallel and different, one if the lines have different slopes, and infinitely many if the two lines are the same. For a line and a circle or parabola, there may be zero, one, or two intersection points. (See Fig. R.5.7.)

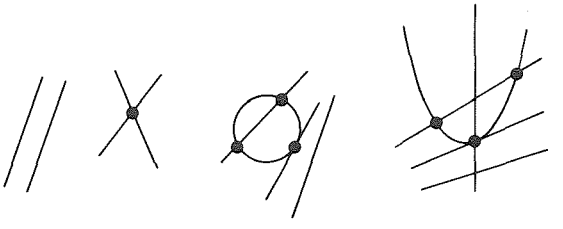

Figure R.5.7. Intersections of some geometric figures.

Example 6 Where do the lines $x + 3y + 8 = 0$ and $y = 3x + 4$ intersect?

Solution To find the intersection point, we solve the simultaneous equations

$$x + 3y + 8 = 0,$$
$$-3x + y - 4 = 0.$$

Multiply the first equation by 3 and add to the second to get the equation $0 + 10y + 20 = 0$, or $y = -2$. Substituting $y = -2$ into the first equation gives $x - 6 + 8 = 0$, or $x = -2$. The intersection point is $(-2, -2)$. ▲

Example 7 Where does the line $x + y = 1$ meet the parabola $y = 2x^2 + 4x + 1$?

Solution We look for pairs (x, y) which satisfy both equations. We may substitute

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$2x^2 + 4x + 1$ for y in the equation of the line to obtain

$$x + 2x^2 + 4x + 1 = 1,$$
$$2x^2 + 5x = 0,$$
$$x(2x + 5) = 0,$$

so $x = 0$ or $-\frac{5}{2}$. We may use either equation to find the corresponding values of y. The linear equation $x + y = 1$ is simpler; it gives $y = 1 - x$, so $y = 1$ when $x = 0$ and $y = \frac{7}{2}$ when $x = -\frac{5}{2}$. Thus the points of intersection are $(0, 1)$ and $(-\frac{5}{2}, \frac{7}{2})$. (See Fig. R.5.8.) ▲

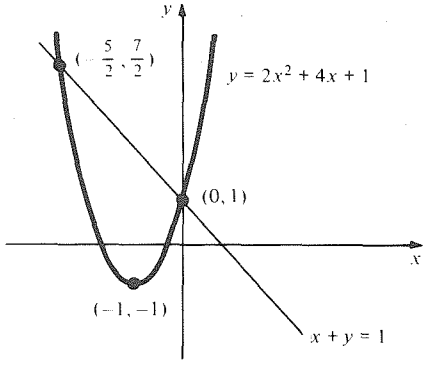

Figure R.5.8. Intersections of the line $x + y = 1$ and the parabola $y = 2x^2 + 4x + 1$ occur at $(0, 1)$ and $(-\frac{5}{2}, \frac{7}{2})$.

Example 8 (a) Where does the line $y = 3x + 4$ intersect the parabola $y = 8x^2$?
(b) For which values of x is $8x^2 < 3x + 4$?

Solution (a) We solve these equations simultaneously:

$$-3x + y - 4 = 0,$$
$$y = 8x^2.$$

Substituting the second equation into the first gives $-3x + 8x^2 - 4 = 0$, so $8x^2 - 3x - 4 = 0$.

By the quadratic formula,

$$x = \frac{3 \pm \sqrt{9 + 4 \cdot 8 \cdot 4}}{16} = \frac{3 \pm \sqrt{137}}{16} \approx \frac{3 \pm 11.705}{16}$$
$$\approx 0.919 \quad \text{and} \quad -0.544.$$

When $x \approx 0.919$, $y \approx 8(0.919)^2 \approx 6.76$. Similarly, when $x \approx -0.544$, $y \approx 2.37$, so the two points of intersection are approximately $(0.919, 6.76)$ and $(-0.544, 2.37)$. (See Fig. R.5.9.) As a check, you may substitute these pairs into the equation $-3x + y - 4 = 0$.

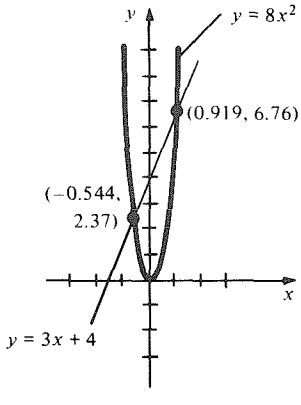

Figure R.5.9. The line $y = 3x + 4$ intersects the parabola $y = 8x^2$ at two points.

If the final quadratic equation had just one root—a double root—there would have been just one point of intersection; if no (real) roots, then no points of intersection.

(b) The inequality is satisfied where the parabola lies below the line, that is, for x in the interval between the x-values of the intersection points. Thus we have $8x^2 < 3x + 4$ whenever

$$x \in \left(\frac{3 - \sqrt{137}}{16}, \frac{3 + \sqrt{137}}{16} \right) \approx (-0.544, 0.919). \blacktriangle$$

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Intersection Points

To find the intersection points of two figures, find pairs (x, y) which simultaneously satisfy the equations describing the two figures.

Exercises for Section R.5

In Exercises 1–4, find the equation of the circle with center at P and radius r. Sketch.

1. $P = (1, 1)$; $r = 3$
2. $P = (-1, 7)$; $r = 5$
3. $P = (0, 5)$; $r = 5$
4. $P = (1, 1)$; $r = 1$

5. Find the equation of the circle whose center is at $(-1, 4)$ and which passes through the point $(0, 1)$. Sketch.
6. Find the equation of the circle with center $(0, 3)$ and which passes through the point $(1, 1)$. Sketch.

Find the center and radius of each of the circles in Exercises 7–10. Sketch.

7. $x^2 + y^2 - 2x + y - \frac{3}{4} = 0$
8. $2x^2 + 2y^2 + 8x + 4y + 3 = 0$
9. $-x^2 - y^2 + 8x - 4y - 11 = 0$
10. $3x^2 + 3y^2 - 6x + 36y - 8 = 0$

Find the equation of the parabola whose vertex is at V and which passes through the point P in Exercises 11–14.

11. $V = (1, 2)$; $P = (0, 1)$
12. $V = (0, 1)$; $P = (1, 2)$
13. $V = (5, 5)$; $P = (0, 0)$
14. $V = (2, 1)$; $P = (1, 4)$

Sketch the graph of each of the parabolas in Exercises 15–18, marking the vertex in each case.

15. $y = x^2 - 4x + 7$
16. $y = -x^2 + 4x - 1$
17. $y = -2x^2 + 8x - 5$
18. $y = 3x^2 + 6x + 2$

Graph each of the equations in Exercises 19–24.

19. $y = -3x^2$
20. $y = -3x^2 + 4$
21. $y = -6x^2 + 8$
22. $y = -3(x + 4)^2 + 4$
23. $y = 4x^2 + 4x + 1$
24. $y = 2(x + 1)^2 - x^2$

Find the points where the pairs of figures described by the equations in Exercises 25–30 intersect. Sketch a graph.

25. $y = -2x + 7$ and $y = 5x + 1$
26. $y = \frac{1}{3}x - 4$ and $y = 2x^2$
27. $y = 5x^2$ and $y = -6x + 7$
28. $x^2 + y^2 - 2y - 3 = 0$ and $y = 3x + 1$
29. $y = 3x^2$ and $y - x + 1 = 0$
30. $y = x^2$ and $x^2 + (y - 1)^2 = 1$

31. What are the possible numbers of points of intersection between two circles? Make a drawing similar to Fig. R.5.7.
32. What are the possible numbers of points of intersection between a circle and a parabola? Make a drawing similar to Fig. R.5.7.

Find the points of intersection between the graphs of each of the pairs of equations in Exercises 33–36. Sketch your answers.

33. $y = 4x^2$ and $x^2 + 2y + y^2 - 3 = 0$
34. $x^2 + 2x + y^2 = 0$ and $x^2 - 2x + y^2 = 0$
35. $y = x^2 + 4x + 5$ and $y = x^2 - 1$
36. $x^2 + (y - 1)^2 = 1$ and $y = -x^2 + 1$

In Exercises 37–40, determine for which x the inequality is true and explain your answer geometrically.

37. $9x^2 < x + 1$
38. $x^2 - 1 < x$
39. $-4x^2 > 2x - 1$
40. $x^2 - 7x + 6 < 0$

R.6 Functions and Graphs

A curve which intersects each vertical line at most once is the graph of a function.

In arithmetic and algebra, we operate with *numbers* (and letters which represent them). The mathematical objects of central interest in calculus are functions. In this section, we review some basic material concerning functions in preparation for their appearance in calculus.

A *function* f on the real-number line is a rule which associates to each real number x a uniquely specified real number written $f(x)$ and pronounced "f of x." Very often, $f(x)$ is given by a formula (such as $f(x) = x^3 + 3x + 2$)

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which tells us how to compute $f(x)$ when x is given. The process of calculating $f(x)$ is called *evaluating* f at x. We call x the *independent variable* or the *argument* of f.

Example 1 If $f(x) = x^3 + 3x + 2$, what is $f(-2)$, $f(2.9)$, $f(q)$?

Solution Substituting -2 for x in the formula defining f, we have $f(-2) = (-2)^3 + 3(-2) + 2 = -8 - 6 + 2 = -12$. Similarly, $f(2.9) = (2.9)^3 + 3(2.9) + 2 = 24.389 + 8.7 + 2 = 35.089$. Finally, substituting q for x in the formula for f gives $f(q) = q^3 + 3q + 2$. ▲

A function need not be given by a single formula, nor does it have to be denoted by the letter f. For instance, we can define a function H as follows:

$$H(x) = 0 \quad \text{if} \quad x < 0 \quad \text{and} \quad H(x) = 1 \quad \text{if} \quad x \geqslant 0.$$

We have a uniquely specified value $H(x)$ for any given x, so H is a function. For example, $H(3) = 1$ since $3 \geqslant 0$; $H(-5) = 0$ since $-5 < 0$; $H(0) = 1$ since $0 \geqslant 0$. Finally, since $|x| \geqslant 0$ no matter what the value of x, we may write $H(|x|) = 1$.

[3] Calculator Discussion

We can think of a function as a *machine* or a *program* in a calculator or computer which yields the output $f(x)$ when we feed in the number x. (See Fig. R.6.1.)

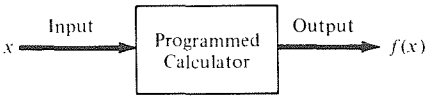

Figure R.6.1. A function on a calculator.

Many pocket calculators have functions built into them. Take, for example, the key labeled x^2. Enter a number x, say, 3.814. Now press the x^2 key and read: 14.546596. The x^2 key represents a function, the "squaring function." Whatever number x is fed in, pressing this key causes the calculator to give x^2 as an output.

We remark that the functions computed by calculators are often only *approximately* equal to the idealized mathematical functions indicated on the keys. For instance, entering 2.000003 and pressing the x^2 key gives the result 4.000012, while squaring 2.000003 by hand gives 4.000012000009. For a more extreme example, let $f(x) = [x^2 - 4.000012] \cdot 10^{13} + 2$. What is $f(2.000003)$? Carrying out the operations by hand gives

$$\begin{aligned} f(2.000003) &= \left[(2.000003)^2 - 4.000012\right] \cdot 10^{13} + 2 \\ &= \left[4.000012000009 - 4.000012\right] \cdot 10^{13} + 2 \\ &= 0.000000000009 \cdot 10^{13} + 2 \\ &= 90 + 2 = 92. \end{aligned}$$

If we used the calculator to square 2.000003, we would obtain $f(2.000003) = 0 \cdot 10^{13} + 2 = 2$, which is nowhere near the correct answer. ▲

Some very simple functions turn out to be quite useful. For instance,

$$f(x) = x$$

defines a perfectly respectable function called the *identity function* ("identity" because if we feed in x we get back the identical number x). Similarly,

[3] This symbol denotes exercises or discussions that may require use of a hand-held calculator.

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$$f(x) = 0$$

is the *zero function* and

$$f(x) = 1$$

is the *constant function* whose value is always 1, no matter what x is fed in.

Some formulas are not defined for all x. For instance, $1/x$ is defined only if $x \neq 0$. With a slightly more general definition, we can still consider $f(x) = 1/x$ as a function.

Definition of a Function

Let D be a set of real numbers. A function f with domain D is a rule which assigns a unique real number $f(x)$ to each number x in D.

If we specify a function by a formula like $f(x) = (x-2)/(x-3)$, its domain may be assumed to consist of all x for which the formula is defined (in this case all $x \neq 3$), unless another domain is explicitly mentioned. If we wish, for example, to consider the squaring function applied only to positive numbers, we would write: "Let f be defined by $f(x) = x^2$ for $x > 0$."

Example 2 (a) What is the domain of $f(x) = 3x/(x^2 - 2x - 3)$? (b) Evaluate $f(1.6)$.

Solution (a) The domain of f consists of all x for which the denominator is not zero. But $x^2 - 2x - 3 = (x-3)(x+1)$ is zero just at $x = 3$ and $x = -1$. Thus, the domain consists of all real numbers except 3 and -1.

(b) $$\begin{aligned} f(1.6) &= 3(1.6)/\left[(1.6)^2 - 2(1.6) - 3\right] \\ &= 4.8/\left[2.56 - 3.2 - 3\right] \\ &= 4.8/(-3.64) \approx -1.32. \end{aligned}$$ ▲

To visualize a function, we can draw its *graph*.

Definition of the Graph

Let f be a function with domain D. The set of all points (x, y) in the plane with x in D and $y = f(x)$ is called the graph of f.

Example 3 (a) Let $f(x) = 3x + 2$. Evaluate $f(-1)$, $f(0)$, $f(1)$, and $f(2.3)$.
(b) Draw the graph of f.

Solution (a) $f(-1) = 3(-1) + 2 = -1$; $f(0) = 3 \cdot 0 + 2 = 2$; $f(1) = 3 \cdot 1 + 2 = 5$; $f(2.3) = 3(2.3) + 2 = 8.9$.

(b) The graph is the set of all (x, y) such that $y = 3x + 2$. This is then just the straight line $y = 3x + 2$. It has y intercept 2 and slope 3, so we can plot it directly (Fig. R.6.2). ▲

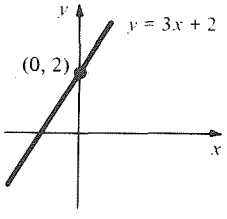

Figure R.6.2. The graph of $f(x) = 3x + 2$ is a line.

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Example 4 Draw the graph of $f(x) = 3x^2$.

Solution The graph of $f(x) = 3x^2$ is just the parabola $y = 3x^2$, drawn in Fig. R.6.3 (see Section R.5). ▲

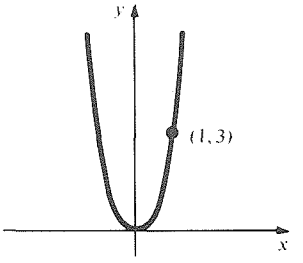

Figure R.6.3. The graph of $f(x) = 3x^2$ is a parabola.

Example 5 Let g be the *absolute value function* defined by $g(x) = |x|$. (The domain consists of all real numbers.) Draw the graph of g.

Solution We begin by choosing various values of x in the domain, computing $g(x)$, and plotting the points $(x, g(x))$. Connecting these points results in the graph shown in Fig. R.6.4. Another approach is to use the definition

$$g(x) = |x| = \begin{cases} x & \text{if } x \geqslant 0, \\ -x & \text{if } x < 0. \end{cases}$$

We observe that the part of the graph of g for $x \geqslant 0$ is a line through $(0, 0)$ with slope 1, while the part for $x < 0$ is a line through $(0, 0)$ with slope -1. It follows that the graph of g is as drawn in Fig. R.6.4. ▲

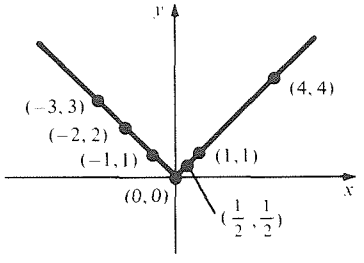

Figure R.6.4. Some points on the graph of $y = |x|$.

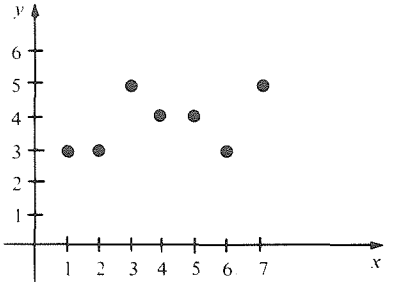

Figure R.6.5. The graph of the number of letters in "one, two, three, four, five, six, seven."

If the domain of a function consists of finitely many points, then the graph consists of isolated dots; there is no line to be filled in. For instance, Fig. R.6.5 shows the graph of a function l whose domain is $\{1, 2, 3, 4, 5, 6, 7\}$ and for which $l(x)$ is the number of letters in the English name for x.

Example 6 Draw the graph $y = \sqrt{x}$.

Solution The domain consists of all $x \geqslant 0$. The graph passes through $(0, 0)$ and may be obtained by plotting a number of points. Alternatively, we can note that for $x \geqslant 0$ and $y \geqslant 0$, $y = \sqrt{x}$ is the same as $y^2 = x$, which is a parabola with the roles of x and y reversed; see Fig. R.6.6. ▲

In plotting a complicated function such as $f(x) = 0.3x^4 - 0.2x^2 - 0.1$, we must be sure to take enough values of x, for we might otherwise miss some important details.

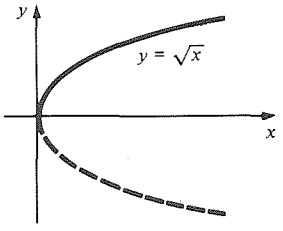

Figure R.6.6. $y = \sqrt{x}$ is half a parabola on its side.

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Example 7 Plot the graph of $f(x) = 0.3x^4 - 0.2x^2 - 0.1$ using

(a) $x = -2, -1, 0, 1, 2$,
(b) x between -2 and 2 at intervals of 0.1.

Solution (a) Choosing $x = -2, -1, 0, 1, 2$ gives the points $(-2, 3.9), (-1, 0), (0, -0.1), (1, 0), (2, 3.9)$ on the graph. (See Fig. R.6.7.) Should we draw a smooth curve through these points? How can we be sure there are no other little bumps in the graph?

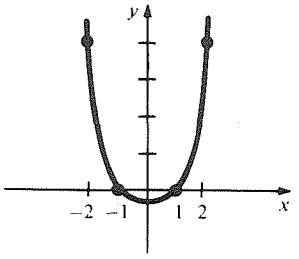

Figure R.6.7. Correct appearance of graph?

(b) To answer this question, we can do some serious calculating:[4] let us plot points on the graph of $f(x) = 0.3x^4 - 0.2x^2 - 0.1$ for values of x at intervals of 0.1 between -2 and 2. If we notice that $f(x)$ is unchanged if x is replaced by $-x$, we can cut the work in half. It is only necessary to calculate $f(x)$ for $x \geqslant 0$, since the values for negative x are the same, and so the graph of f is symmetric about the y axis. The results of this calculation are tabulated and plotted in Fig. R.6.8.

x	$f(x)$	x	$f(x)$	x	$f(x)$	x	$f(x)$
0	−0.10000	0.5	−0.13125	1.0	0.00000	1.5	0.96875
0.1	−0.10197	0.6	−0.13312	1.1	0.09723	1.6	1.35408
0.2	−0.10752	0.7	−0.12597	1.2	0.23408	1.7	1.82763
0.3	−0.11557	0.8	−0.10512	1.3	0.41883	1.8	2.40128
0.4	−0.12432	0.9	−0.06517	1.4	0.66048	1.9	3.08763
						2.0	3.90000

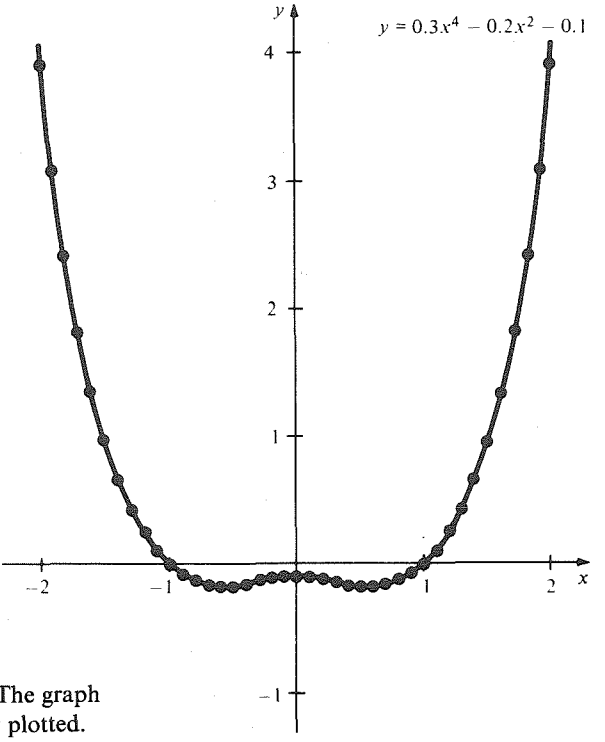

Figure R.6.8. The graph more carefully plotted.

Thus we see that indeed our original guess (Fig. R.6.7) was wrong and that the more refined calculation gives Fig. R.6.8. How can we be sure not to have missed still more bumps and wiggles? By plotting many points we can

[4] Some hints for carrying out these calculations on your calculator are contained in the Student Guide.

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make good guesses but can never know for sure. The calculus we will develop in Chapters 1 to 3 of this book can tell us exactly how many wiggles the graph of a function can have and so will greatly facilitate plotting. ▲

Graphs of functions can have various shapes, but not every set of points in the plane is the graph of a function. Consider, for example, the circle $x^2 + y^2 = 5$ (Fig. R.6.9). If this circle were the graph of a function f, what would $f(2)$ be? Since $(2, 1)$ lies on the circle, we must have $f(2) = 1$. Since $(2, -1)$ also lies on the circle, $f(2)$ should also be equal to -1. But our definition of a function requires that $f(2)$ should have a definite value. Our only escape from this apparent contradiction is the conclusion that *the circle is not the graph of any function*. However, the upper semicircle alone is the graph of $y = \sqrt{5 - x^2}$ and the lower semicircle is the graph of $y = -\sqrt{5 - x^2}$, each with domain $[-\sqrt{5}, \sqrt{5}]$. Thus, while the circle is not a graph, it can be broken into two graphs.

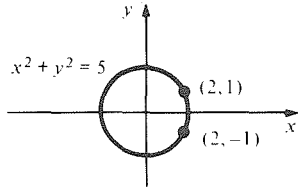

Figure R.6.9. The circle is not the graph of a function.

Example 8 Which straight lines in the plane are graphs of functions?

Solution If a line is not vertical, it has the form $y = mx + b$, so it is the graph of the function $f(x) = mx + b$. (If $m = 0$, the function is a constant function.) A vertical line is not the graph of a function—if the line is $x = a$, then $f(a)$ is not determined since y can take on any value. ▲

There is a test for determining whether a set of points in the plane is the graph of a function. If the number x_0 belongs to the domain of a function f, the vertical line $x = x_0$ intersects the graph of f at the point $(x_0, f(x_0))$ and at no other point. If x_0 does not belong to the domain, (x_0, y) is not on the graph for any value of y, so the vertical line does not intersect the graph at all. Thus we have the following criterion:

Recognizing Graphs of Functions

A set of points in the plane is the graph of a function if and only if every vertical line intersects the set in at most one point.

The domain of the function is the set of x_0 such that the vertical line $x = x_0$ meets the graph.

If C is a set of points satisfying this criterion, we can reconstruct the function f of which C is the graph. For each value x_0 of x, look for a point where the line $x = x_0$ meets C. The y coordinate of this point is $f(x_0)$. If there is no such point, x_0 is not in the domain of f.

Example 9 For each of the sets in Fig. R.6.10:

(i) Tell whether it is the graph of a function.
(ii) If the answer to part (i) is yes, tell whether $x = 3$ is in the domain of the function.
(iii) If the answer to part (ii) is yes, evaluate the function at $x = 3$.

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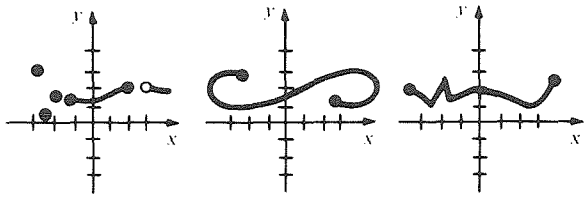

Figure R.6.10. Which curves are graphs of functions?

Solution (a) (i) yes; (ii) no; this is indicated by the white dot.
(b) (i) no; for example, the line $x = 3$ cuts the curve in two points.
(c) (i) yes; (ii) yes; (iii) 1. ▲

Example 10 Which of the curves $y^2 = x$ and $y^3 = x$ is the graph of a function?

Solution We begin with $y^2 = x$. Note that each value of y determines a unique value of x; we plot a few points in Fig. R.6.11.

We see immediately that the vertical line $x = 4$ meets $y^2 = x$ in two points, so $y^2 = x$ cannot be the graph of a function. (The curve $y^2 = x$ is a parabola whose axis of symmetry is horizontal.) Now look at $y^3 = x$. We begin by plotting a few points (Fig. R.6.12).

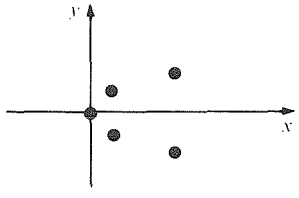

Figure R.6.11. Five points satisfying $y^2 = x$.

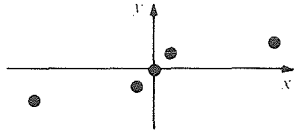

Figure R.6.12. Five points on $y^3 = x$.

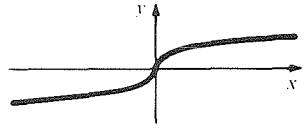

Figure R.6.13. The curve $y^3 = x$.

These points could all lie on the graph of a function. In fact, the full curve $y^3 = x$ appears as in Fig. R.6.13. We see by inspection that the curve intersects each vertical line exactly once, so there is a function f whose graph is the given curve. Since $f(x)$ is a number whose cube is x, f is called the *cube root* function. ▲

Exercises for Section R.6

Evaluate each of the functions in Exercises 1–6 at $x = -1$ and $x = 1$:

1. $f(x) = 5x^2 - 2x$
2. $f(x) = -x^2 + 3x - 5$
3. $f(x) = x^3 - 2x^2 + 1$
4. $f(x) = 4x^2 + x - 2$
5. $f(x) = -x^3 + x^2 - x + 1$
6. $f(x) = (x - 1)^2 + (x + 1)^2 + 2$

Find the domain of each of the functions in Exercises 7–12 and evaluate each function at $x = 10$.

7. $f(x) = \dfrac{x^2}{x - 1}$
8. $f(x) = \dfrac{x^2}{x^2 + 2x - 1}$
9. $f(x) = 5x\sqrt{1 - x^2}$
10. $f(x) = \dfrac{x^2 - 1}{\sqrt{x - 4}}$
11. $f(x) = \dfrac{5x + 2}{x^2 - x - 6}$
12. $f(x) = \dfrac{x}{(x^2 - 2)^2}$

Draw the graphs of the indicated functions in Exercises 13–20.

13. f in Exercise 1
14. f in Exercise 2
15. f in Exercise 4
16. f in Exercise 6
17. $f(x) = (x - 1)^2 + 3$
18. $f(x) = x^2 - 9$
19. $f(x) = 3x + 10$
20. $f(x) = x^2 + 4x + 2$

Draw the graphs of the functions in Exercises 21–24.

21. $f(x) = |x - 1|$
22. $f(x) = 3|x - 2|$
23. $f(x) = \sqrt{x - 1}$

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24. $f(x) = 2\sqrt{x-2}$

Plot 10 points on the graphs of the functions in Exercises 25 and 26.

25. f in Exercise 7
26. f in Exercise 10

Plot the graphs of the functions in Exercises 27 and 28 using the given sets of values for x.

27. $f(x) = 3x^3 - x^2 + 1$:
    (a) at $x = -2, -1, 0, 1, 2$;
    (b) at 0.2 intervals in $[-2, 2]$.
28. $f(x) = 2x^4 - x^3$:
    (a) at $x = -2, -1, 0, 1, 2$;
    (b) at 0.1 intervals in $[-2, 2]$.
29. Which of the curves in Fig. R.6.14 are graphs of functions?

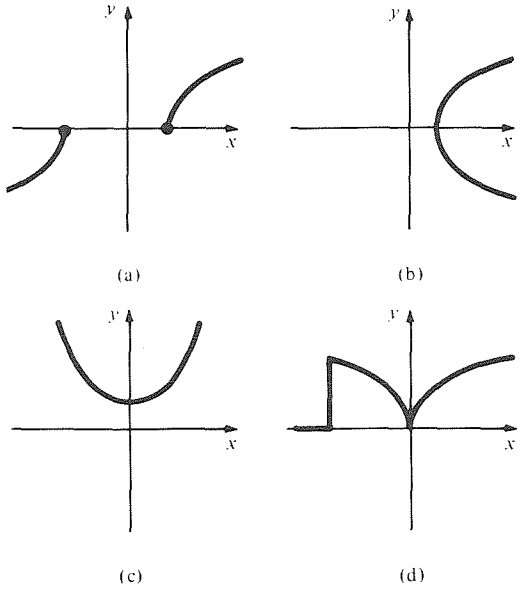

Figure R.6.14. Which curves are graphs of functions?

30. Match the following formulas with the sets in Fig. R.6.15:
    (a) $x + y \geqslant 1$
    (b) $x - y \leqslant 1$
    (c) $y = (x-1)^2$
    (d) $y = \left(\dfrac{x + |x|}{2}\right)^2$
    (e) Not the graph of a function
    (f) None of the above

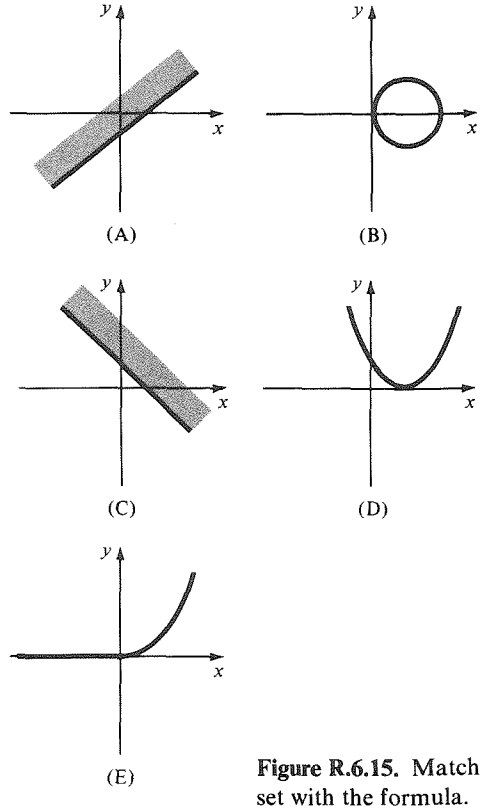

Figure R.6.15. Match the set with the formula.

Tell whether the curve defined by each of the equations in Exercises 31–36 is the graph of a function. If the answer is yes, find the domain of the function.

31. $xy = 1$
32. $x^2 - y^2 = 1$
33. $y = \sqrt{x^2 - 1}$
34. $x + y^2 = 3$
35. $y + x^2 = 3$
36. $x^2 - y^3 = 1$

Review Exercises for Chapter R

In Exercises 1–8, solve for x.

1. $3x + 2 = 0$
2. $x^2 + 2x + 1 = 0$
3. $x(x-1) = 1$
4. $(x-1)^2 - 9 = 0$
5. $(x+1)^2 - (x-1)^2 = 2$
6. $x^2 - 8(x-2) + 10 = 0$
7. $(x-1)^3 = 8$
8. $x^3 + 3x^2 - 3x = 9$

In Exercises 9–16, find all x satisfying the given inequality.

9. $8x + 2 > 0$
10. $10x - 6 < 5$
11. $x^2 - 5 < 0$
12. $x^2 - 5x + 6 < 0$
13. $x^2 - (x-1)^2 > 2$
14. $x^3 > 1$
15. $3x^2 - 7 < 0$
16. $x^2 - 2x + 1 > 0$

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Describe in terms of intervals the set of x satisfying the conditions in Exercises 17–24.

17. $x^2 < 1$
18. $|x - 1| < 2$
19. $x^2 - 2x < 0$
20. $8x - 3 \geqslant 0$
21. $|x - 1|^2 \geqslant 2$
22. $x^2 - 7x + 12 > 0$
23. $x^2 - 11x + 30 < 0$
24. $|x - 3| \leqslant 2$

Find the x satisfying the conditions of Exercises 25–28.

25. $x < 10$ and $x^3 \in (-8, 27)$
26. $-4 \leqslant x < 3$ and $x^2 \geqslant 2$
27. $x \in [5, 9)$ and $20 \leqslant x^2 < 36$
28. $x \in (-4, \frac{3}{4}]$ and $x \in [-\sqrt{2}, 3)$

Find the x satisfying the conditions of Exercises 29–32, and express your answer in terms of intervals.

29. $2(7 - x) \geqslant 1$ or $3x - 22 > 0$.
30. $3x^2 - 7 < 0$ and $x \leqslant 1$.
31. $(2x - 5)(x - 3) > 0$ or $x \in (-5, 1]$, but *not* both.
32. $(x - \frac{7}{5})(3x - 10) \leqslant 0$ and $x^2(5x - 15) > 0$.

Simplify the expressions in Exercises 33–36.

33. $|-8| + 5$
34. $|(a + 1)^2 - 2a - 1|$
35. $|10 - (6 + 12)|$
36. $|-18| + |-2|$

Simplify the expressions in Exercises 37–44.

37. $\sqrt{2} \cdot 2^{-1/2}$
38. $\dfrac{8}{4^{-2}}$
39. $\dfrac{(2 - 3^{1/2})(2 + 3^{1/2})}{2^{-2}}$
40. $\dfrac{(\sqrt{2} - \sqrt{3})(\sqrt{2} + \sqrt{3})\sqrt{2}}{2^{1/2}3^{-2}}$
41. $\dfrac{x^{1/4}\sqrt{x}\, y}{x^{1/2}y^{3/4}}$
42. $(x + y)^{1/4}(\sqrt{x} + 2\sqrt[4]{x}\sqrt[4]{y} + \sqrt{y})^{-1/4}$
43. $\dfrac{1}{\sqrt{x - 1} + \sqrt{x - 8}}$
44. $\dfrac{\sqrt{x}}{\sqrt{x + 2} - \sqrt{x + 5}}$

Find the distance between the pairs of points in Exercises 45–48.

45. $(-1, 1)$ and $(2, 0)$
46. $(2, 0)$ and $(10, 1)$
47. $(5, 5)$ and $(10, 10)$
48. $(100, 100)$ and $(-100, -100)$

Find the equation of the line passing through each of the pairs of points in Exercises 49–52.

49. $(\frac{1}{2}, -1), (7, 3)$
50. $(-1, 6), (1, -6)$
51. $(-2, 3), (1, 0)$
52. $(5, 5), (-3, -2)$

Find the equation of the line passing through the given point P with slope m in Exercises 53–56.

53. $P = (\frac{3}{4}, 13)$, $m = -3$
54. $P = (15, -1)$, $m = 9$
55. $P = (-2, 10)$, $m = \frac{7}{2}$
56. $P = (-9, -5)$, $m = -1$

Find the equations of the straight lines with the given data in Exercises 57–60.

57. Passes through $(1, 1)$ and is perpendicular to the line $5y + 8x = 3$.
58. Passes through $(2, 3)$ and is parallel to the line $y + 7x = 1$.
59. Passes through $(2, 4)$ and is horizontal.
60. Passes through $(-2, -4)$ and is vertical.

Find the equation of the circle with center P and radius r in Exercises 61–64.

61. $P = (12, 5)$, $r = 8$
62. $P = (-9, 3\frac{1}{2})$, $r = 3\frac{1}{2}$
63. $P = (-1, 7)$, $r = 3$
64. $P = (-1, -1)$, $r = 1$

Graph the parabolas in Exercises 65–68.

65. $y = 3x^2 + 4x + 2$
66. $y = 3x^2 - 4x + 2$
67. $y = -2x^2 + 1$
68. $y = -x^2 + 2x$

Find the points where the pairs of graphs described in Exercises 69–72 intersect.

69. $x^2 + y^2 = 4$ and $y = x$
70. $x^2 = y$ and $x = y^2$
71. $y = 3x + 4$ and $y = 3(x + 2)$
72. $2x + 4y = 6$ and $y = x^2 + 1$

Sketch the graphs of the functions in Exercises 73–76.

73. $f(x) = 3|x|$
74. $f(x) = |x| - x$
75. $f(x) = 5x^3 + 1$
76. $f(x) = 1 - x^3$

Plot the graphs of the functions in Exercises 77–80 by evaluating f as indicated.

77. $f(x) = \frac{1}{10}(x^3 - x)$:
    (a) at $x = -2, 0, 2$;
    (b) at intervals of 0.5 in $[-2, 2]$.
78. $f(x) = -x^4 + 3x^2$
    (a) at $x = -2, -1, 0, 1, 2$;
    (b) at intervals of 0.1 in $[-2, 2]$.
79. $f(x) = (x - \frac{1}{2})^{2/3}$
    (a) at $x = -2, -1, 0, 1, 2$;
    (b) at intervals of 0.1 in $[-2, 2]$.
80. $f(x) = x^3 + 1/x^2$
    (a) at $x = -2, -1, -\frac{1}{2}, \frac{1}{2}, 1, 2$;
    (b) at intervals of 0.1 in $[-1, 1]$, $x \neq 0$.

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81. Which of the sets in Figure R.R.1 are graphs of functions?

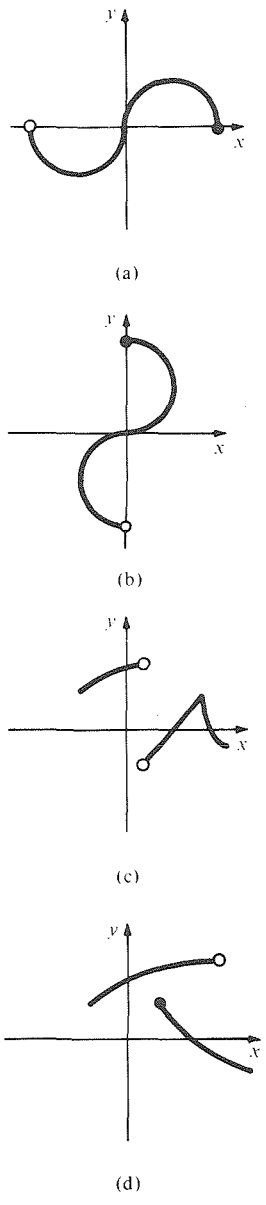

Figure R.R.1. Which sets are graphs of functions?

82. Match the graphs with the formulas.

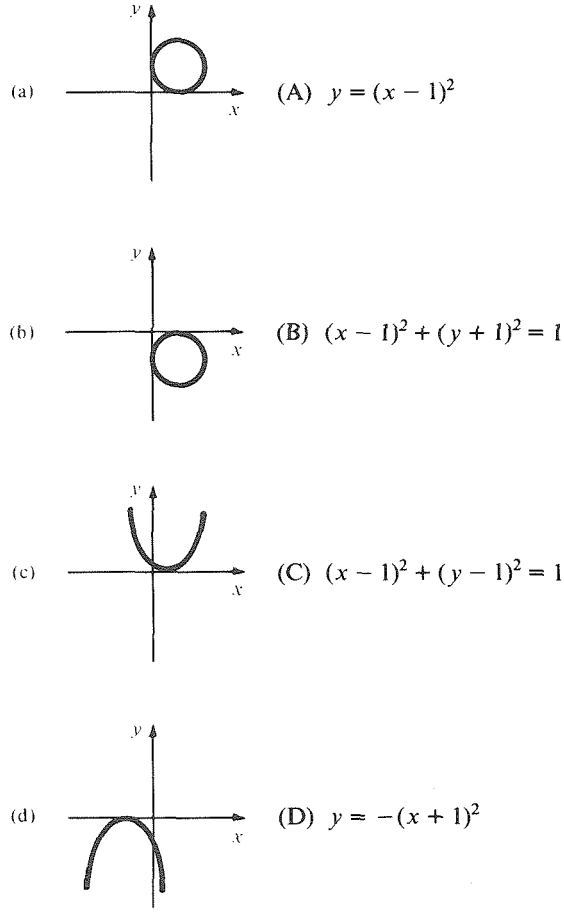

Figure R.R.2. Matching.

★83. (a) If k and l are positive, show that

$$\frac{1}{1 + 1/(k + l)} > \frac{1}{1 + 1/l}.$$

(b) Using the result in (b), show that, if k and l are positive numbers, then $f(k + l) > f(l)$, where $f(x) = x/(x + 1)$.

★84. (a) Prove that a circle and a parabola can intersect in at most four points.

(b) Give examples to show that 0, 1, 2, 3, or 4 intersection points are possible.

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